Unification of one-dimensional Fokker-Planck equations beyond hypergeometrics: Factorizer solution method and eigenvalue schemes

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A one-dimensional Fokker-Planck equation with nonmonotonic exponentially dependent drift and diffusion coefficients is defined by further generalizing a previously studied ''unifying stochastic Markov process.'' The equation, which has six essential parameters, defines and unifies a large class of interdisciplinary relevant stochastic processes, many of them being ''embedded'' as limiting cases. In addition to several known processes that previously have been solved independently, the equation also covers a wide ''interpolating'' variety of different, more general stochastic systems that are characterized by a more complex state dependence of the stochastic forces determining the process. The systems can be driven by additive and/or multiplicative noises. They can have saturating or nonsaturating characteristics and they can have unimodal or bimodal equilibrium distributions. Mathematically, the generalization considered parallels the extension from the Gauss hypergeometric to the Heun differential equation, by adding one more finite regular singularity and its associated confluence possibilities. A previously developed constructive solution method, based upon double integral transforms and contour integral representation, is extended for the actual equation by introducing ''factorizers'' and by using a few of their fundamental properties (compiled in Appendix A). In addition, the equivalent Schrödinger equation and the reflection symmetry principle prove to be important tools for analysis. Fully analytical results including normalization are obtained for the discrete part of the generally mixed spectrum. Only the eigenvalues have to be numerically determined as zeros of a spectral kernel. This kernel generally is unknown, but its zeros are accessible via appropriate, infinite continued fraction based search schemes. The basic role of ''congruence'' in this context is highlighted. For clarity, the simpler standard case corresponding to directly accessible zeros is elaborated first in sufficient detail and the necessary extensions are gradually introduced afterward. The different types of solutions known to exist for Heun's equation eigenvalue problems are identified and are seen to have a ''unified'' structure as well. A small selection of case studies proves ''downward'' compatibility with the previous hypergeometric case and sketches the principles for deriving the limiting results in confluent cases with fully discrete spectra. Possible fields of application are, e.g., population dynamics in biology, noise in nonlinear electronic circuits, chemical and nuclear reaction kinetics, systems with noise-induced transitions or transitions to bimodality, genetics, and neural network stochastics. [S1063-651X(98)08001-5]

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I. INTRODUCTION

There are several good reasons for considering and attempting to solve the Fokker-Planck equation (FPE)

$$\frac{\partial w(y,\tau)}{\partial \tau}=\frac{\partial^2}{\partial y^2}\left(\frac{du^2+fu+g}{au^2+bu+c}w\right)-\frac{\partial}{\partial y}\left(\frac{hu^2+mu+n}{au^2+bu+c}w\right),\quad u(y)=\exp(\rho y),$$

$$y\in[-\infty,+\infty],\quad \tau\in[0,+\infty],\quad a,b,c,d,f,g\geqslant 0. \tag{1}$$

First, the equation is a straightforward generalization of the FPE for a ''unifying'' stochastic Markov process [1] by the addition of $u^2(y)$ terms in the numerators and denominators of the drift and the diffusion coefficients. This unifying FPE has been solved in terms of hypergeometric (Jacobi) functions and was shown to unify a large class of well-known and more recent stochastic processes [1]. In the present extension drift and diffusion still remain bounded functions of y (for finite nonzero parameters), but nonmonotonic behavior now becomes possible and the existence (for specific parameter combinations) of bimodal equilibrium distributions can be anticipated. The generalization considered aims at additionally embedding a number of physically relevant or mathematically important cases as special, limiting, or confluent cases into one single equation. Some known examples are the laser FPE [2], the hyperbolic sine model [3], the FPE for neutron thermalization in a heavy gas moderator [4], and a theoretical stochastic process with spectral accumulation point [5]. Although these cases were solved independently, they all formally belong to the present class, as can be seen either directly as for [3] or after a simple exponential transformation of the state variable for the other examples. In addition to these known cases, which in fact inspired the present extension, the FPE (1) also unifies an impressive variety of generalizations of them and of different, stochastic processes for multidisciplinary use. By comparison with [3], e.g., it can be seen that basic birth-death processes such as the Verhulst process or the hyperbolic sine process can now be extended to account for nonlinear (instead of just linear) population-dependent feedback upon the net birthrate and/or for a state-dependent or even a noisy immigration source (instead of just a constant one). Also, some state dependence of the net birthrate fluctuations can now be modeled. As discussed in [3], the above processes are of multidisciplinary significance and the use of population dynamics terminology is only incidental. In more general terms, because of its structure and parameters, the FPE (1) has the following capabilities. It allows for the modeling of stochastic systems with either additive or multiplicative noise or both (Sec. II). The systems may have saturating (bounded) characteristics or nonsaturating ones if some parameters are allowed to vanish or become infinite (Sec. V). Those systems that are stochastically stable can have a unimodal or a bimodal equilibrium distribution (Sec. III). In view of these possibilities further applications of Eq. (1) can be anticipated in such domains as noisy nonlinear circuits, neural network stochastics, systems with noise-induced transitions or transitions to bimodality, and reaction kinetics, among many others.

A unifying Green's-function solution for Eq. (1) is naturally expected to have continued fraction building blocks. Examples in [2–4] have indeed been solved using some version of the (scalar or matrix) continued fraction method [2], starting with a judicious but somewhat arbitrary choice of base functions. On the other hand, a close connection to hypergeometric functions is expected too because Eq. (1) reduces to the unifying FPE of [1] whenever one power of $u(y)$ (i.e., 0, 1, or 2) is absent everywhere. In a sense, Eq. (1) interpolates between these three hypergeometric cases. The enhanced unification of the FPE (1) carries over to the class of equivalent Schrödinger equations [2]. This equivalence will briefly be discussed in Sec. II, highlighting the type of potentials and the spectral structure for the extended unifying class.

A second reason for considering Eq. (1) is methodological. In order of increasing complexity, the equation is the next test case for a constructive solution method that was used in [6] and further developed in [1]. The method uses double integral transforms and tools from complex analysis for direct construction of the exact analytical Green's function (i.e., the transition probability density function). From this representation, the complete ''spectral package'' can easily be extracted: the discrete eigenvalues and the continua, the eigenfunctions with their normalizing constants, and the weight function (i.e., the normalized steady-state probability density). Because this time the peculiarities of second-order recurrences (or difference equations) come into play, the question to answer is how far analyticity can still be pushed. It will appear that analytical bivariate continued fractions (or more generally ''factorizers''), emerging quite naturally from operator factorization, will extend the applicability of the solution method. The use of the method is not restricted to the actual FPE. It can be succesfully applied in other problems of mathematical or statistical physics, such as wave propagation in nonhomogeneous media, linear magnetohydrodynamic waves in plasma astrophysics, Schrödinger equations, and some higher-dimensional FPE's.

A third aspect of Eq. (1) is the expected spin-off in the field of applied mathematics. The eigenfunctions obtained will realize a unification beyond and including hypergeometric special functions. The ordinary second-order differential equation resulting from classical separation of variables in Eq. (1) and thus defining the eigenvalue problem is reducible to Heun's equation (see Appendix C), which generalizes Gauss's hypergeometric equation by having one more regular singularity. This equation as such never became very popular in mathematical physics, although particular versions of it (mainly confluent, biconfluent, double-confluent, triconfluent, and Lamé's equations) have been considered in several applications. The existing literature is rather old, incomplete, or unreliable (see, e.g., comments in [7]) and only recently [8] became available. In the present paper the solutions are produced in an alternative way, without explicit reference to and even without ''solving'' Heun's equation. Comparison with [8] will allow one to identify the results obtained with known special solutions of Heun's equation, such as Heun's functions and Heun's polynomials. The use of analytical continued fractions in the context of a well-posed physical problem and in the framework of the present solution method will reveal further properties of these interesting mathematical objects.

The contents of the paper are as follows. Section II presents the Fokker-Planck problem in a suitably parametrized form and with reference to the equivalent Schrödinger problem. In Sec. III the FPE is solved by the method of [1]. This necessitates the introduction of factorizers, whose properties are discussed in Appendix A. In their simplest form the factorizers just reduce to forward infinite continued fractions. This ''standard case'' is assumed for simplicity throughout the solution procedure in Sec. III. Section IV discusses the necessary extensions to general factorizers. A small selection of examples is introduced in Sec. V. Section VI summarizes and concludes.

II. THE FPE, STOCHASTIC PROCESS, AND EQUIVALENT SCHRÖDINGER PROBLEM

The fully parametrized form of Eq. (1) with ten parameters is most useful in matching particular applications and especially for deriving limiting cases with zero or infinite parameter values. However, for convenience, the following minimal parametric version with six essential parameters is adopted for the subsequent analysis:

$$\frac{\partial w(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}\left(\frac{v^2+\beta v+\alpha}{\alpha v^2+\gamma v+1}w\right) - 2\frac{\partial}{\partial x}\left(\frac{\delta v^2+\beta\epsilon v+\alpha\mu}{\alpha v^2+\gamma v+1}w\right), \quad v(x)=\exp(x). \tag{2}$$

This version is arrived at by scaling and translation of the independent variables in Eq. (1):

$$x=\rho y+\frac{1}{4}\ln\left(\frac{ad}{cg}\right), \quad x\in[-\infty,+\infty]$$

$$t=\left(\frac{dg}{ac}\right)^{1/2}\rho^2\tau, \quad t\in[0,+\infty] \tag{3}$$

and the parameters are given by

$$\alpha=\left(\frac{ag}{cd}\right)^{1/2},\quad \beta=\frac{f}{d}\left(\frac{ad}{cg}\right)^{1/4},\quad \gamma=\frac{b}{c}\left(\frac{ad}{cg}\right)^{-1/4},$$

$$\delta=\frac{h}{2\rho d},\quad \epsilon=\frac{m}{2\rho f},\quad \mu=\frac{n}{2\rho g}. \tag{4}$$

One notes the important reflection symmetry of Eq. (2):

$$w(x,t;\alpha,\beta,\gamma,\delta,\epsilon,\mu)=w\left(-x,t;\frac{1}{\alpha},\frac{\beta}{\alpha},\frac{\gamma}{\alpha},-\mu,-\epsilon,-\delta\right). \tag{5}$$

The (Stratonovich) stochastic differential equation (SDE) underlying the FPE (2) is given by

$$\dot{x}=f(x)+g(x)\xi(t), \tag{6}$$

where the functions $f(x)$ and $g(x)$ are related to the diffusion and drift of Eq. (2), respectively, by [2,9]

$$g^2(x)=\frac{v^2+\beta v+\alpha}{\alpha v^2+\gamma v+1},\quad v(x)=\exp(x),$$

$$\begin{aligned}f(x)&=2\frac{\delta v^2+\beta\epsilon v+\alpha\mu}{\alpha v^2+\gamma v+1}-g\frac{dg}{dx}\\&=\frac{1}{2}[4\alpha\delta v^4+(4\gamma\delta+4\alpha\beta\epsilon+\alpha\beta-\gamma)v^3\\&\quad+2(2\alpha^2\mu+2\beta\gamma\epsilon+\alpha^2+2\delta-1)v^2\\&\quad+(4\alpha\gamma\mu+4\beta\epsilon+\alpha\gamma-\beta)v+4\alpha\mu]\\&\quad\times(\alpha v^2+\gamma v+1)^{-2}\end{aligned} \tag{7}$$

and $\xi(t)$ represents normalized white noise

$$\langle\xi(t)\rangle=0,\ \langle\xi(t+\tau)\xi(t)\rangle=2\delta(\tau). \tag{8}$$

The stochastic Markov process $\{x(t)\}$ modeled by Eq. (6) is completely characterized by its transition probability density function (PDF) $w(x,t|x_0)$, which is the Green's-function solution of the FPE (2), with

$$w(x,0|x_0)=\delta(x-x_0) \tag{9}$$

and subject to natural boundary conditions

$$w(x,t|x_0),\quad \frac{\partial w}{\partial x}\to 0\ \text{ for } |x|\to\infty. \tag{10}$$

A usual representation of w is in terms of an eigenfunction expansion [2,9,10]

$$w(x,t|x_0)=w_S(x)\left[\sum\int\right]\varphi_k(x)\varphi_k(x_0)e^{-\lambda_k t}, \tag{11}$$

where $w_S(x)$ is the steady-state or first-order PDF (which exists in the stochastically stable case) and the bracketed sum-integral symbol points to the possible existence of a mixed (discrete plus continuous) spectrum of eigenvalues λ_k in general. Some restrictions upon the parameters of Eq. (2) will now be introduced from physical arguments. An unconditionally non-negative diffusion coefficient over $x\in[-\infty,+\infty]$ is ensured by taking

$$\alpha,\beta,\gamma\geqslant 0. \tag{12}$$

The existence of $w_S(x)$ in Eq. (11), i.e., the stochastic stability in probability, additionally demands that

$$\delta<0,\quad \mu>0, \tag{13}$$

which follows by inspection of the limit values $f(\pm\infty)$ in Eq. (7).

The constant diffusion version of the FPE (2)

$$\frac{\partial w(\zeta,t)}{\partial t}=\frac{\partial^2 w}{\partial\zeta^2}-\frac{\partial}{\partial\zeta}[F(\zeta)w] \tag{14}$$

and, correspondingly, the additive noise version of the SDE (6)

$$\dot{\zeta}(t)=\frac{f(x(\zeta))}{g(x(\zeta))}+\xi(t) \tag{15}$$

are obtained by the transformation

$$\zeta(x)=\int_{-\infty}^{x}dx'g^{-1}(x'). \tag{16}$$

These versions are rather academic, as the transformation (16) is noninvertible in general. The drift function in Eq. (14), however, which now is given by

$$F(\zeta)=\frac{f(x(\zeta))}{g(x(\zeta))}, \tag{17}$$

directly allows for the formal construction of the equivalent Schrödinger potential V_S,

$$V_S(\zeta)=\frac{1}{4}F^2+\frac{1}{2}\frac{dF}{d\zeta}=\left[\frac{1}{4}\left(\frac{f}{g}\right)^2+\frac{g}{2}\frac{d}{dx}\left(\frac{f}{g}\right)\right]_{x=x(\zeta)}. \tag{18}$$

As a ratio of two polynomials of degree eight in $\exp[x(\zeta)]$, V_S generally is a bounded potential with asymptotic energy levels

$$V_S(\zeta(+\infty))=\delta^2/\alpha,\quad V_S(\zeta(-\infty))=\alpha\mu^2. \tag{19}$$

Between the energy levels (19), there is a continuum of reflecting states. For higher energies there is a second ''free'' continuum and below these continua at least one bound state exists [if Eq. (13) is satisfied]. Although there are only six essential parameters, V_S models an interesting variety of essentially single-well, double-well, and barrier-well configurations, as appears from a numerical exploration of Eq. (18). Quantitatively, well depths, barrier heights, widths, and relative positions can be varied parametrically over wide ranges. Confining potentials arise for limiting parameter values (see Sec. V D). Quasibound states inside the continuum may exist if a potential well develops between the levels (19) or when a high interwell barrier extends into the continua. For any parameter combination, a plot of V_S [Eq. (18)] as a function of x is a good indicator of the spectral structure to expect. Such a plot is qualitatively similar to a plot of $V_S(\zeta)$, because $\zeta(x)$ is a monotonic function of x for parameters satisfying Eq. (12).

III. SOLUTION OF THE FPE

The solution method of [1] basically consisted in the algebraic construction of a contour integral representation for the Green's-function solution. The classical spectral representation (11) was then obtained by an appropriate contour deformation, which simultaneously generated all components. Application of the same method in the present case seems to be hampered by the peculiarities of second-order recurrences, for which closed-form solutions generally are difficult or impossible to find. With some restrictions, these difficulties can be avoided by the use of factorizers (with continued fractions as a special case). For this and other reasons it seems useful to elaborate the solution procedure in sufficient detail. By twofold transformation of Eq. (2), subject to the initial condition (9) and boundary conditions (10), it is seen that the Fourier-Laplace transform

$$\overline{\eta}(z,p)=\int_{-\infty}^{\infty}dx\ e^{zx}\int_{0}^{\infty}dt\,e^{-pt}\left[\frac{w(x,t|x_0)}{\alpha e^{2x}+\gamma e^{x}+1}\right],$$

$$z=i\omega,\quad \omega\in[-\infty,+\infty], \tag{20}$$

satisfies the functional recurrence equation (FRE) of second order

$$A_0(z,p)\,\overline{\eta}(z,p)+B_0(z,p)\,\overline{\eta}(z+1,p)+C_0(z,p)\,\overline{\eta}(z+2,p)=e^{zx_0}, \tag{21}$$

with coefficients

$$A_0(z,p)=p-\alpha p_0(z),\quad p_0(z)=z^2+2\mu z,$$

$$B_0(z,p)=\gamma p-\beta v_0(z),\quad v_0(z)=z^2+2\epsilon z,$$

$$C_0(z,p)=\alpha p-s_0(z),\quad s_0(z)=z^2+2\delta z. \tag{22}$$

Using the shift operator Δ, defined by

$$\Delta^k G(z)=G(z+k)=\Delta(\Delta^{k-1}G), \tag{23}$$

Eq. (21) becomes

$$(A_0+B_0\Delta+C_0\Delta^2)\,\overline{\eta}(z,p)=e^{zx_0} \tag{24}$$

and is formally solved as

$$\overline{\eta}(z,p)=\left(1+\frac{B_0}{A_0}\Delta+\frac{C_0}{A_0}\Delta^2\right)^{-1}\left(\frac{e^{zx_0}}{A_0}\right). \tag{25}$$

A. Direct inversion

One possible representation for $\overline{\eta}$ results from direct expansion of the inverse operator

$$\overline{\eta}(z,p)=\sum_{k=0}^{\infty}(-1)^k\left(\frac{B_0}{A_0}\Delta+\frac{C_0}{A_0}\Delta^2\right)^k\left(\frac{e^{zx_0}}{A_0}\right)=\frac{e^{zx_0}}{A_0}\,\psi(z,p;x_0), \tag{26}$$

where ψ is a formal power series in e^{x_0},

$$\psi(z,p;x_0)=\sum_{k=0}^{\infty}(-e^{x_0})^k\left(\prod_{j=0}^{k-1}\frac{B_j}{A_{j+1}}\right)T_{(k)}(z,p). \tag{27}$$

The following subscript convention is used

$$F_k(z,p)=F_0(z+k,p)=\Delta^kF_0,$$

$$F_{(k)}(z,p)\neq F_{(0)}(z+k,p). \tag{28}$$

Coefficients $T_{(k)}$ in Eq. (27) follow from the homogeneous second-order FRE in z:

$$T_{(k)}(z)=T_{(k-1)}(z+1)-H_0T_{(k-2)}(z+2),$$

$$H_0(z,p)=\frac{C_0A_1}{B_0B_1},\quad T_{(-1)}=T_{(0)}=1. \tag{29}$$

The first few $T_{(k)}$'s

$$T_{(0)}=T_{(1)}=1,$$

$$T_{(2)}=1-H_0,$$

$$T_{(3)}=1-H_0-H_1,$$

$$T_{(4)}=1-H_0-H_1-H_2+H_0H_2,$$

$$T_{(5)}=1-H_0-H_1-H_2-H_3+H_0H_2+H_0H_3+H_1H_3 \tag{30}$$

illustrate the Euler-Minding sum structure [11] of these coefficients. As such, they are obtainable from the corresponding products

$$S_{(k)}=\prod_{j=0}^{k-2}(1-H_j) \tag{31}$$

by deleting all terms containing at least one pair of consecutive subscripts (e.g., H_0H_1, H_1H_2, and $-H_0H_1H_2$ for $T_{(4)}$). The denominators of the surviving terms are products of distinct B_j $(j=0,1,...,k-1)$ and hence $(\prod_{j=0}^{k-1}B_j)T_{(k)}$ is an entire function of z and p. It follows that $\overline{\eta}(z,p)$ [Eqs. (26)–(27)] is meromorphic in p, having an infinity of simple poles where $A_k=0$, i.e., for

$$p=\alpha p_k(z)=\alpha p_0(z+k),\quad k=0,1,\ldots,\infty. \tag{32}$$

As, moreover, $\overline{\eta}$ vanishes for $|p|\to\infty$, a partial fraction (or ''critical parts'') representation preparing for Laplace inversion is possible [12]:

$$\overline{\eta}(z,p)=\sum_{k=0}^{\infty}\frac{r_{(k)}(z)}{p-\alpha p_k(z)}. \tag{33}$$

Residues $r_{(k)}$ can be obtained according to their definition

$$r_{(k)}(z)=\lim_{p\to\alpha p_k}[A_k(z,p)\,\overline{\eta(z,p)}], \tag{34}$$

but they also satisfy the recurrence

$$r_{(k)}(z)+\left[\frac{B_0}{A_0}\right]_k r_{(k-1)}(z+1)+\left[\frac{C_0}{A_0}\right]_k r_{(k-2)}(z+2)=0, \tag{35}$$

following from substitution of Eq. (33) in Eq. (24) and application of Eq. (34). The $[\ ]_k$ notation is used to indicate that $p=\alpha p_k(z)$ has been substituted throughout. Equation (35) is solved in terms of the $T_{(k)}$'s of Eq. (29):

$$r_{(k)}(z)=(-1)^k\left[T_{(k)}\prod_{j=0}^{k-1}\frac{B_j}{A_j}\right]_k r_{(0)}(z+k),\quad k=1,2,...,\infty, \tag{36}$$

which starts up from

$$r_{(0)}(z)=\lim_{p\to\alpha p_0}(A_0\overline{\eta})=e^{zx_0}\psi[z,\alpha p_0(z);x_0]. \tag{37}$$

Laplace inversion of Eq. (33), using Eqs. (36) and (37), yields

$$\begin{aligned}\eta(z,t)&=\mathcal{L}^{-1}[\overline{\eta(z,p)};p\to t]\\&=\sum_{k=0}^{\infty}(-1)^k e^{\alpha p_k(z)t+(z+k)x_0}\\&\quad\times\left[T_{(k)}\prod_{j=0}^{k-1}\frac{B_j}{A_j}\right]_k\psi[z+k,\alpha p_k(z);x_0].\end{aligned} \tag{38}$$

Up to this point the track of [1] has been closely followed. For the next step, i.e., the transition from the sum (38) to a contour integral representation, the dependence of $T_{(k)}$ upon k must be explicited. The Euler-Minding sum gives just a recipe, but no closed analytical form. Additional tools are necessary, and these can be introduced most naturally by restarting the analysis from an alternative inversion of the operator in Eq. (25).

B. Factorized inversion: Continued fractions

The operator in Eqs. (24) or (25) can be factorized

$$\left(1+\frac{B_0}{A_0}\Delta+\frac{C_0}{A_0}\Delta^2\right)=\left(1+\frac{R_0}{A_0}\Delta\right)\left(1+\frac{S_0}{A_0}\Delta\right), \tag{39}$$

where the factorizers $R_0(z,p)$ and $S_0(z,p)$ solve the pair of coupled nonlinear first-order FRE's

$$\begin{aligned}R_0+S_0&=B_0,\\R_0S_1&=C_0A_1=D_0.\end{aligned} \tag{40}$$

The solutions that will be retained in this section are directly obtained by forward iteration of Eq. (40). They are the bivariate infinite continued fractions (ICF's)

$$\begin{aligned}R_0=P_0(z,p)&=\frac{D_0}{B_1-P_1}=\frac{D_0|}{|B_1}-\frac{D_1|}{|B_2}-\cdots-\frac{D_k|}{|B_{k+1}}-\cdots\\&=\cfrac{D_0}{B_1-\cfrac{D_1}{B_2-\cfrac{D_2}{B_3-\cdots}}},\\S_0=Q_0(z,p)&=B_0-P_0=B_0-\frac{D_0|}{|B_1}-\cdots-\frac{D_k|}{|B_{k+1}}-\cdots,\end{aligned} \tag{41}$$

where the notational convention of [11] is used. These functions exist and are computable under the simple condition

$$\beta^2>4\alpha, \tag{42}$$

which is derived from Perron's more general convergence criterion [11], and henceforth will be accepted as an additional parameter constraint. Useful properties of factorizers (ICF's and others) can be found in Appendix A and the use of factorizers other than P_0,Q_0 will be discussed in Sec. IV. Inversion of the operator (39) now proceeds in two steps

$$\begin{aligned}\overline{\eta(z,p)}&=\left(1+\frac{Q_0}{A_0}\Delta\right)^{-1}\left(1+\frac{P_0}{A_0}\Delta\right)^{-1}\left(\frac{e^{zx_0}}{A_0}\right)\\&=\left(1+\frac{Q_0}{A_0}\Delta\right)^{-1}\left[\frac{e^{zx_0}}{A_0}\chi_0(z,p;x_0)\right],\end{aligned} \tag{43}$$

where χ_0 generalizes the hypergeometric ${}_3F_2$ expression obtained by inversion of the first-order shift operator in [1],

$$\begin{aligned}\chi_0(z,p;x_0)&=\sum_{k=0}^{\infty}(-e^{x_0})^k\prod_{j=0}^{k-1}\frac{P_j}{A_{j+1}}\\&=\sum_{k=0}^{\infty}(-e^{x_0})^k\prod_{j=0}^{k-1}\frac{C_j}{Q_{j+1}}.\end{aligned} \tag{44}$$

Further, with $\chi_k=\chi_0(z+k)$,

$$\overline{\eta(z,p)}=\frac{e^{zx_0}}{A_0}\sum_{k=0}^{\infty}(-e^{x_0})^k\chi_k\prod_{j=0}^{k-1}\frac{Q_j}{A_{j+1}}, \tag{45}$$

where the sum gives an alternative representation of ψ [Eq. (27)]. By equating like powers of e^{x_0} in both series, a symmetrical form for the $T_{(k)}$'s is obtained:

$$U_{(k)}=\left(\prod_{j=0}^{k-1}B_j\right)T_{(k)}=\sum_{m=0}^{k}\left(\prod_{j=0}^{m\text{-}1}Q_j\right)\left(\prod_{j=m}^{k-1}P_j\right). \tag{46}$$

In the residues (36) or in $\eta(z,t)$ [Eq. (38)] the above expression is needed only for $p=\alpha p_k(z)$ and thereby reduces to a simple product

$$[U_{(k)}]_k=\left[\left(\prod_{j=0}^{k-1}B_j\right)T_{(k)}\right]_k=\left[\prod_{j=0}^{k-1}Q_j\right]_k \tag{47}$$

because [see Eq. (41)]

$$P_{k-1}(z,\alpha p_k)=\left[\frac{D_{k-1}}{Q_k}\right]_k=\left[\frac{C_{k-1}A_k}{Q_k}\right]_k=0 \tag{48}$$

since [see Eq. (22)]

$$A_k(z,\alpha p_k)\equiv 0. \tag{49}$$

Hereby, it is tacitly assumed that the zero $p=\alpha p_1$ of D_0 in Eq. (41) is a zero of the ICF P_0 indeed, i.e., $Q_1(z,\alpha p_1)\neq 0$ (see, however, Appendix A and Sec. III J 2 for a discussion on ''congruence'').

C. Explicit k dependence

The k dependence of the term in large square brackets in $\eta(z,t)$ [Eq. (38)] can now be made explicit. For the product of A_j's one has, using Eqs. (22) and (32),

$$\left[\prod_{j=0}^{k-1} A_j\right]_k=\prod_{j=0}^{k-1} A_j[z,\alpha p_k(z)]=\alpha^k k!\frac{\Gamma(2z+2\mu+2k)}{\Gamma(2z+2\mu+k)}. \tag{50}$$

For the other factors there is a formal representation, using Eq. (47) and the result (A21) of Appendix A:

$$[U_{(k)}]_k=\prod_{j=0}^{k-1} Q_j(z,\alpha p_k(z))=\frac{M(z,\alpha p_k(z))}{M(z+k,\alpha p_k(z))}. \tag{51}$$

Some properties of the generally unknown function $M(z,p)$ are discussed in Appendix A. Here it suffices to mention the following. (i) M is entire in z and basically has exactly the zeros $z=\rho_m(p)$, $m=0,1,...,$ of the factorizer Q_0. If these where known, then M could be constructed, using well-known theorems of complex function theory (e.g., Hadamard's infinite product representation [13]). (ii) M solves a homogeneous second-order difference equation and hence is only determined up to a period-1 function. (iii) M generalizes the product of inverse Γ functions occurring in [1,6], where the structure (51) was self-evident however. Exactly this structure is the essential point and, even with M unavailable, further progress becomes possible.

D. Contour integral representation

Considering Eq. (38) as a sum of residues generated by an appropriately chosen function, the following integral representation is arrived at:

$$\eta(z,t)=\frac{1}{2\pi i}\int_C dq\; e^{\alpha p_0(q)t+qx_0}\alpha^{z-q}\Gamma(z-q)\frac{\Gamma(z+q+2\mu)}{\Gamma(2q+2\mu)}\times\frac{M(z,\alpha p_0(q))}{M(q,\alpha p_0(q))}H(q,z)\psi(q,\alpha p_0(q);x_0). \tag{52}$$

The complex integration variable

$$q=z+k \tag{53}$$

replaces the integer summation index k in Eq. (38), and Eqs. (50) and (51) have been substituted. Contour C in the complex q plane runs clockwise around the poles of the summator function $\Gamma(z-q)$ [see Eq. (59)] without enclosing other singularities of the integrand. As in [1], the function $H(q,z)$ is introduced as a degree of freedom in the integral representation. Other than some analyticity demands, it should only satisfy the condition

$$H(z+k,z)=1,\quad k=0,1,...,\infty, \tag{54}$$

in order to preserve the equivalence of Eqs. (52) and (38). Clearly, an expression like

$$H(q,z)=\frac{m(q,f(q))}{m(z,f(q))}, \tag{55}$$

where m is of period 1 in its first argument, satisfies Eq. (54) and is a possible choice. Equivalently, H may be taken as unity and the m's can be considered as originating from the indeterminacy of M. In order to obtain now a Fourier-transformed eigenfunction expansion from Eq. (52), the ''eigenvalue'' $[-\alpha p_0(q)]$ figuring in the time exponential must take real non-negative values, as the problem formulated by Eqs. (2) and (10) is Hermitian [2]. To achieve this, the original contour C must move towards the eigenvalue locus L, which is found from :

$$\alpha p_0(q)=\alpha[(q+\mu)^2-\mu^2]=-\lambda,\quad \mathrm{Im}(\lambda)=0,\quad \lambda\geqslant 0 \tag{56}$$

and consists of a cross shape in the q plane: a vertical line

$$\mathrm{Re}(q)=-\mu \tag{57}$$

and a horizontal segment

$$\mathrm{Im}(q)=0,\quad -2\mu\leqslant \mathrm{Re}(q)\leqslant 0. \tag{58}$$

The intended contour deformation is possible if the integrand of Eq. (52) is free of singularities between C and L.

E. Singularity analysis

By construction of Eq. (52), the poles of the summator $\Gamma(z-q)$, i.e., the points

$$q_k=z+k,\quad k=0,1,...,\infty, \tag{59}$$

are all to the right of C. Similarly, the poles of $\Gamma(z+q+2\mu)$, i.e.,

$$q_k=-z-2\mu-k,\quad z=i\omega,\quad k=0,1,...,\infty, \tag{60}$$

are all to the left of L [Eqs. (57) and (58)]. These pole series do not interfere with the contour deformation. Other singularities of the integrand in Eq. (52) only originate from M, which thus contains complete information about the spectral structure of the solution and hence could be termed a ''spectral kernel'' (SK). As M is generally unknown, analyticity seems to cease here. However, the physical context of the problem and especially the strong analogy with the fully analytically developed solution in [1] allow one to anticipate and even prove a few essential properties of M. Although the subsequent analysis necessarily is of a formal and abstract nature, conclusive results will eventually be obtained.

First, it should be noted that the p dependence of $M(z,p)$ exclusively enters via the zeros $a^{\pm}(p)$, $b^{\pm}(p)$, and $c^{\pm}(p)$

of the quadratic polynomials in z (see Appendix B). These ''parameters'' have branch points in p and so generally do M and its z-wise zeros $\rho_m(p)$ (although M may be single valued if the different branches of the multivalued zero functions are symmetrically represented; see further). Therefore, in the q plane, the set of zeros of

$$M(q,\alpha p_0(q))=0 \tag{61}$$

can be thought of as being partitionable into subsets, according to the branch of the ρ_m's where they originate from. The SK M must have at least one ''good'' or ''physical'' branch of zeros q_k that will produce the discrete eigenvalues as poles of the integrand, with the discrete eigenfunctions as residues (see Sec. III F). These zeros result from a formal ''good'' branch equation such as

$$q=\rho_g(\alpha p_0(q)) \tag{62}$$

and must lie on the discrete spectrum part of the eigenvalue locus L $(-\mu\leqslant q\leqslant 0)$ or to the left of it. In the presently assumed standard case [where the ICF's P_0,Q_0 (41) are the appropriate factorizers and Eq. (48) holds] all zeros of M nicely show up in Q_0 (see Appendix A) and the good q_k values can be found by a numerical search for the zeros of

$$Q_0(q,\alpha p_0(q))=0 \tag{63}$$

on the real axis interval (67) (see further).

The SK must have branch points on the locus L, acting as separation points between the different parts of the spectrum. As shown in Appendix B, these branch points are

$$q_A^{\pm}=-\mu,\quad q_C^{\pm}=-\mu\pm\left(\mu^2-\frac{\delta^2}{\alpha^2}\right)^{1/2}, \tag{64}$$

where the locus intersection $q=-\mu$ appears as a degenerate pair of coinciding branch points. In the present case, $M(z,p)$ is single valued in p because P_0 and Q_0 are. The branch points (64) are latent and must be activated by annihilation of ''bad'' zero branches.

The SK must have such ''bad'' or nonphysical zeros, with some of them lying between C and L (incidentally upon L). These obstruct the contour unfolding from C to L and they must be annihilated by incorporating suitable periodic m factors into M [or by H (55)]. If bad zeros q_k originate from a nonphysical branch $\rho_b(p)$ of the zero functions, say,

$$q_k=\rho_b(\alpha p_0(q_k)), \tag{65}$$

then one can formally take

$$m(z,p)=e^{2\pi i[z-\rho_b(p)]}-1. \tag{66}$$

This function is periodic in z and $m(q,\alpha p_0(q))$ has simple zeros at the q_k of the bad branch (65). Inclusion of this m not only annihilates the bad zeros of M but also destroys the original symmetry of M with respect to the distinct zero function branches. Branch points (64) on locus L get activated and the new ''cleaned'' SK, $M(q,\alpha p_0(q))$ (the same symbol is kept), is multivalued in q. Branch cuts must be introduced to isolate a single-valued branch of M in the q plane, which now becomes the principal, or physical, Riemann sheet in which the contour can be unfolded.

In summary, one expects on physical grounds the final true SK to have a number $N+1$ of good zeros q_k (at least $q_0=0$) on the discrete spectrum section of L,

$$\mathrm{Re}(q_c^{\pm})\leqslant q_k\leqslant 0; \tag{67}$$

no other zeros in the half plane $\mathrm{Re}(q)\geqslant-\mu$, i.e., to the right of L; the branch points (64); and appropriate asymptotics for large q $[\mathrm{Re}(q)\geqslant-\mu]$ such that distant contour integral contributions vanish. It is further assumed that this M exists and that its physical zeros q_k have been obtained numerically via the ICF equation (63).

F. Contour deformation: Spectral structure

Rewriting the integral representation (52) compactly as

$$\eta(z,t)=\frac{1}{2\pi i}\int_C dq\; U(q,z,t,x_0) \tag{68}$$

and deforming C so as to cover the locus L [Eqs. (57) and (58)], one obtains

$$\eta(z,t)=\sum_{k=0}^{N}\mathrm{Res}_{q=q_k}(U)+\frac{1}{2\pi i}\int_B dq\; U+\frac{1}{2\pi i}\int_V dq\; U. \tag{69}$$

The sum of residues originates from the good poles $q_0,q_1,\ldots,q_N$ of the integrand that cross the contour during unfolding. The B integral runs either around the right half of a horizontal branch cut $(\alpha^2\mu^2>\delta^2)$ or adjacent to a vertical cut $(\alpha^2\mu^2\leqslant\delta^2)$. Accordingly, the V integral covers the entire vertical locus line or the remaining part of it. The same qualitative figures as in [1] apply. As in [1], a spectral structure with bound states, a ''reflecting'' continuum and a ''free'' continuum is apparent. The discrete spectrum will be calculated in the following subsection. Continuum contributions will not be further elaborated.

G. Discrete spectrum results: Steady-state PDF

The discrete spectrum contribution is obtained by calculation of the residues at the poles of U in Eq. (68); the sum in Eq. (69) becomes

$$[\eta(z,t)]_{\mathrm{discrete}}=\sum_{k=0}^{N} e^{-\lambda_k t}\alpha^{z-q_k}\frac{\Gamma(z+q_k+2\mu)\Gamma(z-q_k)}{\Gamma(2q_k+2\mu)}\times R_{(k)}M(z,-\lambda_k)e^{q_k x_0}\psi(q_k,-\lambda_k;x_0), \tag{70}$$

where

$$\lambda_k=-\alpha p_0(q_k)=-\alpha[(q_k+\mu)^2-\mu^2] \tag{71}$$

and $R_{(k)}$ formally denotes the residue at the pole q_k of the inverse SK,

$$R_{(k)}=\lim_{q\to q_k}\frac{q-q_k}{M(q,\alpha p_0(q))}=\left(\frac{dM(q,\alpha p_0(q))}{dq}\right)^{-1}_{q=q_k}=\left[\frac{\partial M(z,p)}{\partial z}+\frac{\partial M}{\partial p}\frac{\partial p}{\partial z}\right]^{-1}_{p=\alpha p_0(z),z=q_k}. \tag{72}$$

Using the symbolic eigenfunction expansion (11) for the Green's function and the definition (20) of $\overline{\eta(z,p)}$, it is seen that the kth term of the sum (70) corresponds to the Fourier transform

$$\mathcal{F}\left(\frac{w_S(x)\varphi_k(x)}{\alpha e^{2x}+\gamma e^x+1}\varphi_k(x_0)e^{-\lambda_k t},x\to z\right)=e^{-\lambda_k t}\Phi_k(z)\varphi_k(x_0). \tag{73}$$

By identifying the t, x_0, and z dependences, one finds the following results.

(i) The λ_k as given by Eq. (71) are the discrete eigenvalues indeed.

(ii) The eigenfunctions φ_k are given by

$$\varphi_k(x)=N_k e^{q_k x}\psi(q_k,-\lambda_k;x). \tag{74}$$

N_k is an unknown normalization constant and ψ is given by Eq. (27) or, alternatively, by the sum in Eq. (45):

$$\psi(z,p;x)=\sum_{m=0}^{\infty}(-e^x)^m\chi_m(z,p;x)\left(\prod_{j=0}^{m\text{-}1}\frac{Q_j}{A_{j+1}}\right), \tag{75}$$

with χ_0 from Eq. (44) and $\chi_m=\chi_0(z+m)$. For the presently assumed standard case, where q_k solves Eq. (63), the eigenfunctions simply reduce to

$$\begin{aligned}\varphi_k(x)&=N_k e^{q_k x}\chi_0(q_k,-\lambda_k;x)\\&=N_k e^{q_k x}\sum_{m=0}^{\infty}(-e^x)^m\left[\prod_{j=0}^{m\text{-}1}\frac{C_j(q_k,-\lambda_k)}{Q_{j+1}(q_k,-\lambda_k)}\right].\end{aligned} \tag{76}$$

This type of eigenfunction can be identified with the special "Heun function" solution of the Heun equation [8]. The necessary analytic continuation for these functions will be derived in Sec. III H.

(iii) The z-dependent factor gives for Φ_k the equation

$$\begin{aligned}\Phi_k(z)&=\mathcal{F}\left(\frac{w_s(x)\varphi_k(x)}{\alpha e^{2x}+\gamma e^x+1},x\to z\right)\\&=\alpha^{z-q_k}\frac{\Gamma(z+q_k+2\mu)\Gamma(z-q_k)}{\Gamma(2q_k+2\mu)}\frac{R_{(k)}}{N_k}M(z,-\lambda_k).\end{aligned} \tag{77}$$

First using Eq. (77) for $k=0$ [i.e., $q_0=0,\varphi_0(x)=N_0=1$], one has

$$\begin{aligned}\Phi_0(z)&=\mathcal{F}\left(\frac{w_s(x)}{\alpha e^{2x}+\gamma e^x+1},x\to z\right)\\&=R_{(0)}\alpha^z\frac{\Gamma(z+2\mu)\Gamma(z)}{\Gamma(2\mu)}M(z,0),\end{aligned} \tag{78}$$

which basically would allow for the determination of the steady-state PDF $w_S(x)$ if $M(z,0)$ and $R_{(0)}$ were available. It is possible to solve the reduced version ($p=0$) of the general equation (A22) for $M(z,0)$:

$$\begin{aligned}&\alpha z(z+1)(z+2\delta)(z+2\mu+1)M(z+2,0)\\&\quad+\beta z(z+2\epsilon)M(z+1,0)+M(z,0)=0,\end{aligned} \tag{79}$$

but it is easier to solve directly the time-independent FPE [i.e., Eq. (2) with $\partial w/\partial t=0$] for $w_S(x)$. The results are

$$\begin{aligned}w_s(x)&=N_s(\alpha e^{2x}+\gamma e^x+1)e^{2\mu x}(e^x-\rho^+)^{\nu^+-1}\\&\quad\times(e^x-\rho^-)^{\nu^--1},\end{aligned} \tag{80}$$

with

$$\rho^{\pm}=\frac{1}{2}[-\beta\pm(\beta^2-4\alpha)^{1/2}] \tag{81}$$

being the roots of the quadratic equation

$$\rho^2+\beta\rho+\alpha=0 \tag{82}$$

that determines two of the four singularities of the FPE (in the e^x plane). These roots are seen to be real under the same condition (42) that ensures convergence of the ICF's. They are negative and $|\rho^+|<|\rho^-|$. The exponents $\nu^{\pm}$ are given by

$$\nu^{\pm}=\frac{2}{1-a^{\pm1}}[\delta+a^{\pm1}\mu-(1+a^{\pm1})\epsilon],\quad a=\frac{\rho^-}{\rho^+} \tag{83}$$

and satisfy the simple relations

$$\nu^++\nu^-=2(\delta-\mu),\quad a\nu^++\nu^-=2(a+1)(\epsilon-\mu). \tag{84}$$

Under conditions (13), $w_S(\pm\infty)=0$ and can be normalized. For calculating N_S, the Fourier-transform (78) is a useful intermediate result

$$\begin{aligned}\Phi_0(z)&=N_S\int_{-\infty}^{+\infty}dx\;e^{(z+2\mu)x}(e^x-\rho^+)^{\nu^+-1}(e^x-\rho^-)^{\nu^--1}\\&=N_S\Lambda(z),\end{aligned} \tag{85}$$

with, from [14,15] and Appendix D,

$$\begin{aligned}\Lambda(z)&=(-\rho^+)^{z+2\mu+\nu^+-1}\\&\quad\times(-\rho^-)^{\nu^--1}\frac{\Gamma(z+2\mu)\Gamma(2-2\delta-z)}{\Gamma(2\mu-2\delta+2)}\\&\quad\times{}_2F_1\left(z+2\mu,1-\nu^-;2(\mu-\delta+1);1-\frac{\rho^+}{\rho^-}\right).\end{aligned} \tag{86}$$

From $\Phi_0(z)$ one builds the steady-state characteristic function

$$\theta_S(z)=\mathcal{F}(w_S(x),x\to z)=\alpha\Phi_0(z+2)+\gamma\Phi_0(z+1)+\Phi_0(z) \tag{87}$$

and normalization of θ_S yields N_S,

$$\theta_S(0)=1=N_S[\alpha\Lambda(2)+\gamma\Lambda(1)+\Lambda(0)]. \tag{88}$$

Subsequently, $\theta_s(z)$ can be used, e.g., to calculate steady-state moments such as

$$\mu_k = \langle e^{kx} \rangle = \theta_S(k). \tag{89}$$

It is straightforward to verify that the ''equivalent Fokker-Planck potential'' [i.e., minus the logarithm of $w_S(x)$ (80)] is either a monostable or a bistable potential well, with linear asymptotes. A mathematical side result from Eq. (78) is

$$M(z,0) = \frac{N_S}{R_{(0)}} \Lambda(z) \alpha^{-z} \frac{\Gamma(2\mu)}{\Gamma(z+2\mu)\Gamma(z)}, \tag{90}$$

which solves Eq. (79) indeed and yields a closed form (apparently uncatalogued in [16]) for the univariate ICF $Q_0(z,0)$:

$$Q_0(z,0) = \frac{M(z,0)}{M(z+1,0)} = (-\rho^-) z (1-2\delta-z) \frac{{}_2F_1\left(z+2\mu, 1-\nu^-; 2(\mu-\delta+1); 1-\frac{\rho^+}{\rho^-}\right)}{{}_2F_1\left(z+2\mu+1, 1-\nu^-; 2(\mu-\delta+1); 1-\frac{\rho^+}{\rho^-}\right)}. \tag{91}$$

H. Analytic continuation of the discrete eigenfunctions

The present solution method delivers the eigenfunctions as power series in e^x, converging around the singularity at $e^x = 0$, up to the nearest singular point at $e^x = \rho^+$. [Actually, according to [8], the eigenvalue condition (63) implies ''superconvergence'' up to $e^x = \rho^-$. This does not invalidate the following analysis where it is allowed to replace ρ^+ by ρ^- throughout.] One has in general [see Eqs. (74)–(76)]

$$\varphi(x) = N_k e^{q_k x} \sum_{m=0}^{\infty} a_m(q_k)(-e^x)^m, \quad |e^x| < |\rho^+|, \tag{92}$$

with specifically for the standard case

$$a_m(q_k) = \prod_{j=0}^{m\text{-}1} \frac{C_j(q_k, -\lambda_k)}{Q_{j+1}(q_k, -\lambda_k)} = \prod_{j=0}^{m\text{-}1} \frac{P_j(q_k, -\lambda_k)}{A_{j+1}(q_k, -\lambda_k)},$$

$$a_0 = 1. \tag{93}$$

Generalizing Euler's transformation [17,18], one looks for a series in the new variable

$$X(x) = \frac{e^x}{e^x - \rho^+}, \tag{94}$$

which results in the analytic continuation

$$\varphi_k(x) = N_k e^{q_k x} \left(1 - \frac{e^x}{\rho^+}\right)^{-\sigma} \sum_{m=0}^{\infty} b_m(q_k) X^m,$$

$$b_m(q_k) = (\sigma)_m \sum_{l=0}^{m} \frac{(\rho^+)^l a_l(q_k)}{(m-l)!(\sigma)_l}, \quad b_0 = 1, \tag{95}$$

where σ can be chosen for convenience. The simple Euler transformation is obtained for $\sigma = 1$:

$$b_m = \sum_{j=0}^{m} \binom{m}{j} (\rho^+)^j a_j, \tag{96}$$

but another obvious choice is to identify σ with one of the factors of $C_0(q_k, -\lambda_k)$ [see Eq. (B8)],

$$\sigma = \sigma^{\pm}(q_k) = q_k - c^{\pm}(-\lambda_k) = q_k + \delta \mp (\delta^2 - \alpha\lambda_k)^{1/2} \tag{97}$$

such that the b_m's in Eq. (95) reduce to

$$b_m = (\sigma^{\pm})_m \sum_{l=0}^{m} \frac{(-\rho^+)^l (\sigma^{\mp})_l}{(m-l)! \prod_{j=1}^{l} Q_j}. \tag{98}$$

This yields the Euler continuations that are usual in the theory of Heun's differential equation [8,19] and these b_m's satisfy a certain second-order recurrence that will not be derived here. Observing the transformation of the singular points at $e^x = 0$, $\rho^{\pm}$, and ∞ under Eq. (94), it follows that the series in Eq. (95) converges for $|X| < 1$, i.e., for

$$\mathrm{Re}(e^x) > \frac{\rho^+}{2}, \tag{99}$$

such that this representation of φ_k is appropriate for the entire physical domain $x \in [-\infty, +\infty]$. Other continuations will be derived in Sec. IV C.

Returning to Eq. (77), the Fourier transform can now be evaluated by use of $w_S(x)$ [Eq. (80)] and the continuation (95) for $\varphi_k(x)$, all integrals being of the type in Appendix D:

$$\Phi_k(z) = N_S N_k (-\rho^+)^{z+2\mu+q_k+\nu^+-1} (-\rho^-)^{\nu^--1} \times \Gamma(\sigma + 2 - 2\delta - q_k - z) S(z, q_k), \tag{100}$$

where

$$S(z,q_k) = \sum_{m=0}^{\infty} b_m \frac{\Gamma(z+2\mu+q_k+m)}{\Gamma(\sigma-2\delta+2\mu+2+m)} {}_2F_1\left(z+2\mu+q_k + m, 1-\nu^-; \sigma+2-2\delta+2\mu+m; 1-\frac{\rho^+}{\rho^-}\right). \tag{101}$$

This result will allow for the analytical calculation of the normalization N_k of the eigenfunctions $\varphi_k(x)$.

I. Normalization of the discrete eigenfunctions

Substitution of the result (100) into Eq. (77) and solving for N_k yields

$$N_k^2=\frac{(-\rho^+)^{1-z-2\mu-q_k-\nu^+}(-\rho^-)^{1-\nu^-}\alpha^{z-q_k}\Gamma(z+q_k+2\mu)\Gamma(z-q_k)}{N_S S(z,q_k)\Gamma(\sigma+2-2\delta-q_k-z)\Gamma(2q_k+2\mu)}R_{(k)}M(z,-\lambda_k), \tag{102}$$

where $R_{(k)}M(z,-\lambda_k)$ is still unknown. The right-hand side of Eq. (102) must be z independent. It may be evaluated by taking $z=q_k$ and resolving the resulting indeterminacy as follows. In the neighborhood of $z=q_k$ one has

$$M(z,-\lambda_k)=(z-q_k)\left(\frac{\partial M}{\partial z}\right)_{z=q_k,p=-\lambda_k}+O(z-q_k)^2 \tag{103}$$

such that, with $R_{(k)}$ from Eq. (72),

$$R_{(k)}M(z,-\lambda_k)=\frac{(z-q_k)[1+O(z-q_k)]}{1+2\alpha(q_k+\mu)\left(\frac{\partial M}{\partial p}\Big/\frac{\partial M}{\partial z}\right)_{z=q_k,p=-\lambda_k}}; \tag{104}$$

The ratio of unknown partial derivatives in Eq. (104) is expressible in terms of the known ICF $Q_0(z,p)$:

$$V(q_k,-\lambda_k)=\left(\frac{\partial M}{\partial p}\Big/\frac{\partial M}{\partial z}\right)_{z=q_k,p=-\lambda_k}=\left(\frac{\partial Q_0}{\partial p}\Big/\frac{\partial Q_0}{\partial z}\right)_{z=q_k,p=-\lambda_k}, \tag{105}$$

as easily follows from partial differentiation of Eq. (A21). Derivatives of Q_0 are fully analytically determined by recurrence, e.g., [see Eq. (A18)],

$$\frac{\partial Q_0}{\partial p}=\frac{\partial B_0}{\partial p}-\frac{1}{Q_1}\frac{\partial D_0}{\partial p}+\frac{D_0}{Q_1^2}\frac{\partial Q_1}{\partial p}=\left(\gamma-\frac{\alpha A_1+C_0}{Q_1}\right)+\frac{D_0}{Q_1^2}\left[\left(\gamma-\frac{\alpha A_2+C_1}{Q_2}\right)+\cdots, \tag{106}$$

and similarly for $\partial Q_0/\partial z$. They may, however, be obtained numerically as well, e.g., during the numerical search for the q_k values in Eq. (63). Setting now $z=q_k$ in Eq. (102), the normalization is obtained

$$N_k^2=\frac{(-\rho^+)^{1-2q_k-2\mu-\nu^+}(-\rho^-)^{1-\nu^-}}{N_S[1+2\alpha(q_k+\mu)V(q_k,-\lambda_k)]\Gamma(\sigma+2-2\delta-2q_k)S(q_k,q_k)}, \tag{107}$$

with N_S from Eq. (88) and S from Eq. (101). This completes the discrete spectrum results.

J. Searching for physical eigenvalues

The indirect access to the physical zeros of the generally unknown SK function $M(q,\alpha p_0(q))$ via the ICF $Q_0(q,\alpha p_0(q))$ may be hampered by some specific problems.

1. Bad zeros

Even if the search is restricted to the negative real axis segment (67) in the q plane, bad zeros incidentally lying on this part of the locus may be picked up too. A criterion for acceptance or rejection of the zeros is necessary. One possibility is to write Eq. (63) as

$$Q_0(q,\alpha p_0(q))=B_0(q,\alpha p_0(q))-P_0(q,\alpha p_0(q))=0 \tag{108}$$

and to solve this ''quadratic'' equation for q_k, which yields a semiexplicit formula

$$q_k=\frac{1}{\beta-\alpha\gamma}\{(\alpha\gamma\mu-\beta\epsilon)\pm[(\alpha\gamma\mu-\beta\epsilon)^2+(\alpha\gamma-\beta)P_0(q_k,\alpha p_0(q_k))]^{1/2}\},$$
$$\mathrm{Re}[\ ]^{1/2}\geqslant 0. \tag{109}$$

Preliminary numerical evidence shows that physical q_k values satisfy Eq. (109) taken with the same sign as the one necessary for $q_0=0$ to be a root (the physical zero $q_0=0$ selects the physical branch, as in [1]). A more substantial criterion, however, follows from spectral invariance under reflection, i.e., under the ''overbar'' transformation (B5). A physical zero q_k must have a partner $\overline{q_k}$ solving $\overline{Q_0}(\overline{q_k},\overline{\alpha p_0}(\overline{q_k}))=0$ and generating the same eigenvalue [see Eq. (B7)]. It appears (again there is no proof) that a nonphysical q_k (or $\overline{q_k}$) does not have such a partner.

2. Congruent zeros

The basics of congruence are discussed in Appendix A. For some values of the second variable p, the z-wise zeros of $M(z,p)$ may become left congruent and as such will not show up in the ICF Q_0 anymore. Their existence, however, is still signaled by the vanishing of Q_0 at the starting zero of the congruent series, which is a ''moved'' free zero [for the free zeros d_i see Eqs. (A2) and (B8)]. Hence one has to look for λ values in the discrete spectrum range

$$0 \leqslant \lambda \leqslant \min(\alpha\mu^2, \delta^2/\alpha) \tag{110}$$

for which

$$Q_0(d_i(-\lambda)+1,-\lambda)=0. \tag{111}$$

If this condition for ''left congruence with respect to d_i'' is satisfied, then M vanishes at all left-congruent positions

$$M(d_i(-\lambda)+1-k,-\lambda)=0; \quad k=0,1,...,\infty. \tag{112}$$

This equation coincides with the eigenvalue condition (61) if for some $k=N_c$

$$d_i(-\lambda)+1-N_c=q, \tag{113}$$

where q corresponds to λ as usual via Eq. (71). It is seen that congruence defines a two-parameter (λ,N_c) [or (q,N_c)] eigenvalue problem. Because of Eqs. (113) and (111) it follows that for a congruent q value

$$D_{N_c-1}(q,-\lambda)=0, \quad Q_{N_c}(q,-\lambda)=0 \tag{114}$$

such that all ICF's P_j and Q_j for $j<N_c$ become indeterminate. This breakdown in the calculation of $Q_0(q,-\lambda)$ (during the standard case search e.g.) may be used as an incidental congruence detector, but the better strategy is first to calculate explicitly analytically the candidate $q(N_c)$ values from Eq. (113), and then to check the $Q_{N_c}=0$ condition. For the eigenfunctions corresponding to congruent eigenvalues, one may envisage a complicated de l'Hôpital limit for D_{N_c-1}/Q_{N_c}, but a direct recurrent calculation of their coefficients by Eq. (29) is more appropriate. One notes the useful equivalents for Eqs. (29) or (46):

$$T_{(k)}(z)=T_{(k-1)}(z)-H_{k-2}T_{(k-2)}(z), \tag{115a}$$

$$U_{(k)}(z)=B_{k-1}U_{(k-1)}(z)-D_{k-2}U_{(k-2)}(z), \tag{115b}$$

$$U_{(k)}(z)=U_{(k-1)}(z)P_{k-1}+\prod_{j=0}^{k-1} Q_j\,. \tag{115c}$$

Equation (115c) can be used from $k=N_c+1$ on and then simply reduces to

$$U_{(k)}=U_{(k-1)}P_{k-1}, \tag{116}$$

showing the corresponding coefficients to have the same pure product structure as in the standard case Heun function. This type of solution could be termed ''semiminimal.''

On substitution of the free zeros, one finds the following. (i) For $d_i(p)\equiv a^+(p)-1$, Eq. (113) explicitly becomes

$$a^+(-\lambda(q))-N_c=q-N_c=q, \tag{117}$$

which is identically satisfied for all q if $N_c=0$. As, moreover, Eq. (114) reduces to Eq. (63), one just retrieves the q values of the standard case, which thus appears as a ''degenerate'' congruent case. All these q values are acceptable with $N_c=0$ and the problem effectively becomes a one-parameter problem.

(ii) With $d_i(p)\equiv c^-(p)$, Eq. (113) can be written as

$$c^-(-\lambda(q))-q=c^-(-\lambda)-a^+(-\lambda)=-(\overline{q}+q)=N_c-1 \tag{118}$$

and solved for $q(N_c)$,

$$q(N_c)=\frac{1}{1-\alpha^2}\{(\alpha^2\mu-\delta-N_c+1)\pm[(\alpha^2\mu-\delta-N_c+1)^2 +(N_c-1)(2\delta+N_c-1)(\alpha^2-1)]^{1/2}\},$$
$$\mathrm{Re}(q+\delta+N_c-1)<0. \tag{119}$$

If such a q satisfies $Q_{N_c}(q,-\lambda(q))=0$, then a congruent eigenvalue has been found. To be a physical eigenvalue, the same λ must be retrievable from the overbar transformed [see Eq. (B5)] search procedure. Because only Eq. (118) is invariant under this transformation, left congruence with respect to c^- is the only nondegenerate congruence scheme that can produce physical eigenvalues and the options $d_i \equiv a^- -1$ or $d_i\equiv c^+$ need not be considered anymore. The overbar version of Eq. (114) becomes [see Eq. (160)]

$$\overline{Q}_{N_c}(\overline{q},-\lambda)=\frac{1}{\alpha}P^*_{-N_c}(-\overline{q},-\lambda)=\frac{1}{\alpha}P^*_0(q-1,-\lambda)=0, \tag{120}$$

which shows a physical congruent λ also to cause right congruence with respect to a^+-1. Although Eq. (118) coincides with the necessary condition (upon the Frobenius exponent at ∞; see Appendic C) for the existence of Heun's ''polynomials'' [8], the additional transcendental equation $Q_{N_c}=0$ seems to be just a condition for extended convergence [8] and not for truncation of the solution series. This generally yields ''transcendental Heun functions'' for the congruent eigenfunctions and these need to be analytically continued (see Sec. III H). Incidentally, however, the solution series may truncate to a Heun polynomial, converging over the entire e^x plane. The exact condition for this to occur is not further investigated here (classically [8], this is an N_cth degree polynomial equation for the accessory parameter b defined in Appendix C).

3. Close-to-congruent cases

The possibility of congruent zero series, as well as the existence of three hypergeometric limiting cases (see Secs. I and V) implies that many FPE parameter combinations give rise to a close-to-congruent situation, where at least two zeros of M become almost unitarily spaced. When this happens, the q_k zero values can get extremely close to poles of Q_0 and when the zero-pole separation falls below numerical resolution both disappear, even when theoretically congruence is not perfect. To remedy this problem, one may access the zeros of M via the product of ICF's

$$\prod_{j=0}^{N} Q_j=\frac{M_0}{M_{N+1}} \tag{121}$$

instead of just via $Q_0=M_0/M_1$. By taking for N a sufficiently large positive integer, the ''shadowing'' poles are shifted and the ''hidden'' zeros can be resolved. Alternatively, the expression

$$\prod_{j=0}^{N}\left(1-\frac{P_j}{B_j}\right)=\left(\prod_{j=0}^{N}B_j\right)^{-1}\frac{M_0}{M_{N+1}} \qquad (122)$$

having the same zeros as Eq. (121) can be used and is often much better conditioned from a numerical point of view. Basically, useful N values must be determined by trial and error. As such, the eigenvalue problem appears as a ''weak'' version of the strictly two-parameter (q,N_c) eigenvalue problem arising in the case of exact congruence (see Sec. III J 2).

4. Failure of the standard case approach

A simple numerical exploration of Eq. (63) and/or Eq. (111) reveals that for some parameter sets no eigenvalues can be found at all (except $\lambda_0=0$), although the equivalent Schrödinger potential (18) is strongly confining or very close to a potential that is known to have bound states (e.g., the case in [3]). This is not unexpected, as it is known [2,20] that sometimes a dominant solution of the recurrence is necessary for constructing the physical eigenfunctions, while the forward ICF Q_0 only gives access to the minimal solution M. Moreover, backward ICF's may be necessary for generating the eigenvalues. This opens the gate for the general factorizers derived in Appendix A and for some related extensions.

IV. EXTENSIONS

A. Introducing general factorizers

The developments in the present section strongly depend upon the material introduced in Appendix A, to which the reader is referred for more details. Replacing the forward ICF's P_0 and Q_0, as special solutions of Eq. (40), by general factorizers R_0 and S_0, Eq. (46) for the coefficients becomes

$$U_{(k)}=\sum_{m=0}^{k}\left(\prod_{j=0}^{m-1}S_j\right)\left(\prod_{j=m}^{k-1}R_j\right) \qquad (123)$$

or after factorizing out the first term of the sum

$$U_{(k)}=\left(\prod_{j=0}^{k-1}R_j\right)\sum_{m=0}^{k}\left(\prod_{j=0}^{m-1}\frac{S_j}{R_j}\right). \qquad (124)$$

The infinite sum

$$L(z)=\sum_{m=0}^{\infty}\left(\prod_{j=0}^{m-1}\frac{S_j}{R_j}\right) \qquad (125)$$

converges under the condition

$$\lim_{k\to\infty}\left|\frac{S_k}{R_k}\right|<1, \qquad (126)$$

which can be shown to hold if the K_0 function composing R_0 and S_0 behaves asymptotically dominant in the right half z plane. When satisfied, one has

$$U_{(k)}=\left(\prod_{j=0}^{k-1}R_j\right)L(z)-\left(\prod_{j=0}^{k-1}S_j\right)\frac{S_k}{R_k}L(z+k+1) \qquad (127)$$

and, in addition, an ICF representation for the series [11,16]

$$L(z)=\frac{1|}{|1}-\frac{\frac{S_0}{R_0}\Big|}{\Big|1+\frac{S_0}{R_0}}-\frac{\frac{S_1}{R_1}\Big|}{\Big|1+\frac{S_1}{R_1}}-\cdots . \qquad (128)$$

By an obvious equivalence transformation and the use of Eq. (40), this reduces to

$$L(z)=\frac{Q_0}{Q_0-S_0} \qquad (129)$$

such that

$$U_{(k)}=\left(\prod_{j=0}^{k-1}R_j\right)\frac{Q_0}{Q_0-S_0}-\left(\prod_{j=0}^{k-1}S_j\right)\frac{S_k}{Q_k-S_k}; \qquad (130)$$

Substitution herein of Eq. (A10) for the general factorizers R_0,S_0, Eq. (A26) for K_0, Eq. (A21) for P_0,Q_0, and use of the zero mover properties (A27) and (A28) gives, after some algebra,

$$U_{(k)}=\frac{M_0M_{k+1}^*-\left(\prod_{j=0}^{k}D_{j-1}\right)M_0^*M_{k+1}}{M_0M_1^*-D_{-1}M_0^*M_1}. \qquad (131)$$

Property (A25), also used in the above calculation, states that the denominator of Eq. (131) is periodic in z:

$$\mathcal{D}_0(z,p)=M_0M_1^*-D_{-1}M_0^*M_1=\mathcal{D}_1(z,p)=\mathcal{D}_0(z+1,p) \qquad (132)$$

such that many alternative forms in terms of the ICF's become possible, e.g.,

$$U_{(k)}=\frac{Q_0\left(\prod_{j=0}^{k-1}P_j^*\right)-Q_0^*\left(\prod_{j=0}^{k-1}P_j\right)}{Q_0-Q_0^*}=\frac{\prod_{j=0}^{k}Q_j-\prod_{j=0}^{k}Q_j^*}{Q_k-Q_k^*}. \qquad (133)$$

It is seen that the use of general factorizers reduces, at the level of the coefficients $U_{(k)}$, to the combined use of forward and backward ICF's or their associated minimal solutions M_0 and M_0^*. This result is completely independent of any specific choice related to R_0,S_0,K_0. All free zeros d_i have an equivalent role and the constant distribution factor f [see Eqs. (A4) and (A5)] drops out as well as all arbitrary periodics $\omega(z)$. The alternative of a left dominant K_0, i.e.,

$$\lim_{k\to-\infty}\left|\frac{R_k}{S_k}\right|<1, \qquad (134)$$

instead of Eq. (126), of course leads to the same result via a formal interchange of forward and backward ICF's. Proceeding further as in the standard case, it is seen that the ratio of M functions in the contour integral representation (52) is now replaced by

$$T(z,q,\lambda)=\frac{M_0(z,-\lambda)M_1^*(q,-\lambda)}{\mathcal{D}_0(z,-\lambda)}, \tag{135}$$

with $\lambda=\lambda(q)=-\alpha p_0(q)$ as in Eq. (71).

B. Eigenvalues

The result (135) as a spectral kernel is puzzling, as it seems to imply the eigenvalue equation

$$\mathcal{D}_0(z,-\lambda)=0, \tag{136}$$

which normally should produce z-dependent eigenvalues. Again $\mathcal{D}_0$ is unknown, but at least its noncongruent zeros are accessible via the ICF's

$$Q_0-Q_0^*=P_0^*-P_0=\frac{M_0}{M_1}-D_{-1}\frac{M_0^*}{M_1^*}=\frac{\mathcal{D}_0}{M_1M_1^*} \tag{137}$$

such that a numerical exploration of Eq. (136) is possible. This has revealed the following facts.

(a) For certain FPE parameter sets, λ_k values can be found in the physical range (110) that satisfy Eq. (136) independently of the value of z. The latter, however, sensibly determines the numerical resolvability of these zeros and can be optimized to avoid congruence.

(b) Other λ_k values for the same parameter set, or λ_k's for other parameter combinations, exhibit a cyclic behavior indeed when z is gradually varied to $z+1$. The observed behavior may be just a back and forth movement along the real λ axis or even a disappearance (possibly becoming complex or obscured by congruence) during part of the cycle.

The above phenomena can be understood from the structure

$$\mathcal{D}_0(z,p)=F(p)\{e^{2\pi i[z-\pi^+(p)]}-1\}\{e^{2\pi i[z-\pi^-(p)]}-1\}, \tag{138}$$

which can be derived by a tedious asymptotic analysis of $\mathcal{D}_0$ [Eq. (132)] at the upper and lower ends of a fundamental period strip in the z plane, using the technique described in [21,22]. Unfortunately, the p-dependent functions F and $\pi^\pm$ remain unknown and there seems to be no possibility to determine them. The present context, however, allows one to conclude that the $\pi^\pm(p)$ must be the branches of a two-valued function, having known branch points (see Sec. III E and Appendix B). They are determined up to an additive integer only. The λ_k values of type (a) result from $F(-\lambda)=0$ and necessarily are physical eigenvalues as they cannot be suppressed. Type (b) $\lambda_k(z)$ values manifestly are nonphysical, but because they originate from a periodical factor in Eq. (138) their annihilation is possible. It suffices to redefine the H function [see Eqs. (52), (54), (55)] as

$$H(q,z)=\frac{m(z,-\lambda(q))}{m(q,-\lambda(q))}, \tag{139}$$

where $m(z,p)$ is a suitable entire periodic function of z having the bad zero branch of $\mathcal{D}_0(z,p)$ [say $\pi^+(p)$]. Starting with the simplest possible choice

$$m(z,p)=e^{2\pi i[z-\pi^+(p)]}-1, \tag{140}$$

it is seen that the nonphysical z-dependent poles of the integrand are destroyed, the integrand gets activated branch points (as in Sec. III E), and, moreover, the eigenvalue condition now effectively becomes

$$m(q,-\lambda(q))=0 \tag{141}$$

or, in view of Eq. (140) and with N any integer,

$$q-\pi^+[-\lambda(q)]=-N. \tag{142}$$

If π^+ were known, then this equation could be solved for the physically possible (q_k,N_k) combinations (note that generally $N_k\neq k$; k just labels the eigenvalues). With unknown π^+, solutions of Eq. (142) have to be found as solutions of

$$\mathcal{D}_0(q+N,-\lambda(q))=0 \tag{143}$$

via the ICF's [see Eq. (137)] and again one has to be aware of bad, congruent, and close-to-congruent zeros. This eigenvalue equation could have been obtained also by formally using $\mathcal{D}_0(z+k+N,-\lambda)$ instead of $\mathcal{D}_0(z,-\lambda)$ in deriving Eq. (135) and by substitution of $z+k=q$ here as well when switching to the integral representation. This procedure, however, is less general than the annihilation scheme, as will be seen further.

On setting $N=0$, the standard case search of Sec. III is reproduced because Eq. (135) reduces to

$$T(z,q,\lambda)=\frac{M_0(z,-\lambda(q))}{M_0(q,-\lambda(q))}. \tag{144}$$

If this search fails to yield the complete discrete spectrum (see Sec. III J) then other N values may be considered, accessing Eq. (143) via

$$Q_0(q+N,-\lambda(q))-Q_0^*(q+N,-\lambda(q))$$
$$=Q_N(q,-\lambda)-Q_N^*(q,-\lambda)=0, \tag{145}$$

where either Q_N (for $N<0$) or Q_N^* (for $N\geqslant 0$) truncates to a finite continued fraction because $D_{-(N+1)}(q+N,-\lambda(q))\equiv 0$.

From Eq. (137) it easily follows that, e.g., for $N\geqslant 0$,

$$Q_N-Q_N^*=(Q_0-Q_0^*)\prod_{j=1}^{N}\frac{Q_j}{P_{j-1}^*}, \tag{146}$$

which already shows that searching with Eq. (145) gives a superior resolution of close-to-congruent cases in much the same way as the heuristically introduced formulas (121) or (122). The full power of the extension to general factorizers, however, becomes clear when dealing with exact congruences. One observes the following facts.

(i) In addition to left congruence in the forward ICF's as discussed in Sec. III J 2 and detected by

$$Q_0(c^-(-\lambda)+1,-\lambda)=0, \tag{147}$$

there now also is the possibility for right congruence (with respect to $d_j \equiv c^-$) in the backward ICF's, detectable by

$$P_0^*(c^-(-\lambda),-\lambda)=0. \tag{148}$$

(ii) In any case of congruence, $\mathcal{D}_0$ acquires the full infinite series of zeros

$$\mathcal{D}_0(c^-(-\lambda)+k,-\lambda)=0, \quad k=-\infty,\ldots,-1,0,1,\ldots,+\infty, \tag{149}$$

as follows from the periodicity (132).

(iii) A λ value causing congruence is an eigenvalue if it falls in the physical range (110) for discrete eigenvalues and if its corresponding q value [see Eq. (71)] satisfies

$$c^-(-\lambda(q))+1-N_c=q \tag{150}$$

for some integer N_c such that Eq. (149) coincides with the eigenvalue condition (143). In contrast to Eq. (113), N_c is not restricted here to non-negative values. The analytical solution of Eq. (150) for $q(N_c)$ is still given by Eq. (119). For eigenvalues resulting from left congruence (147) in position N_c, Eq. (114) and the resulting indeterminacy of P_j, Q_j for $j<N_c$ still apply. Similarly, for a right congruence (148) one has $D_{N_c-1}(q,-\lambda)=0$, $\quad P^*_{N_c-1}(q,-\lambda)=0$, and P_j^*, Q_j^* indeterminate for $j \geqslant N_c$.

(iv) Also in contrast to the standard case, ''congruent'' q values generally do not require special treatment. They do show up as zeros in searches (145) with $Q_N-Q_N^*=0$, just as standard case q values in Eq. (63), provided N is properly chosen. This can be understood from Eq. (137). When, e.g., M is left congruent, then $Q_0-Q_0^*$ still displays the rightmost zeros c^-+k for $k\geqslant 1$ of the fully infinite series (149) and so does $Q_N-Q_N^*$ for $k\geqslant -N+1$. Likewise, on right congruence in M^*, $Q_N-Q_N^*$ displays the leftmost zeros for $k<-N+1$. Substituting $k=1-N_c$ in the above inequalities defines the values of N that ensure visibility of a zero in $Q_N-Q_N^*$, whenever the candidate $q(N_c)$ truly becomes a congruent q value. These values of N also are the appropriate ones for avoiding the indeterminate ICF's on congruence.

(v) The simultaneous occurrence of left and right congruence [conditions (147) and (148)] cannot be excluded *a priori*. The q values resulting from such a double congruence would stay invisible for any N and in fact the access to $\mathcal{D}_0$ via $Q_N-Q_N^*$ even fails because there is always one indeterminate ICF. The values $q(N_c)$, however, are known from Eq. (119).

One further generalization of the above eigenvalue search scheme is obtained if in Eq. (140) the bad (z-dependent) zero annihilator period is reduced to N_p^{-1}:

$$m(z,p)=e^{2\pi i N_p[z-\pi^+(p)]}-1, \quad N_p=1,2,\ldots . \tag{151}$$

This eventually leads to a search procedure for the zeros of

$$\mathcal{D}_0\left(q+\frac{N}{N_p},-\lambda(q)\right)=\mathcal{D}_m(q+\nu_n,-\lambda(q))=0,$$

$$\frac{N}{N_p}=m+\nu_n, \quad m=\operatorname{int}\left(\frac{N}{N_p}\right),$$

$$\nu_n=\frac{n}{N_p}, \quad n=0,1,\ldots,N_p-1 \tag{152}$$

via the appropriate ICF differences

$$Q_j(q+\nu_n,-\lambda)-Q_j^*(q+\nu_n,-\lambda)=0, \quad j=\ldots,-1,0,1,\ldots, \tag{153}$$

as can easily be verified. This procedure covers the standard case ($N=0$ or $m=\nu_n=0$), the previous extension ($N=mN_p$ or $\nu_n=0$), and N_p-1 additional eigenvalue schemes corresponding to the possible (nonzero) values of the rational number ν_n. Schemes with ν_n and $-\nu_{N_p-n}=\nu_n-1$ are equivalent. The integer m (or j) is physically irrelevant and just determines the numerical accessibility of the eigenvalues. The parameter ν_n, however, codetermines their very existence and their value. Parallel to Eq. (152), the congruence relation generalizes to

$$c^-(-\lambda(q))+1-N_c=q+\nu_n \tag{154}$$

such that

$$D_{N_c-1}(q+\nu_n,-\lambda)=0,$$

$$Q_0(c^-+1,-\lambda)=Q_{N_c}(q+\nu_n,-\lambda)=0$$

or

$$P_0^*(c^-,-\lambda)=P^*_{N_c-1}(q+\nu_n,-\lambda)=0 \tag{155}$$

Again, Eq. (155) causes indeterminacy of some ICF's, but all ICF's figuring in the eigenfunctions have the arguments (q, $-\lambda$) and stay perfectly determinate when the eigenvalue originates from a $\nu_n\neq 0$ scheme. Because N_p explicitly enters into the solution, it must be uniquely assignable for any given FPE parameter set. A decisive method has not been found yet, and further research is guided by some facts and conjectures.

(i) The coexistence of two distinct eigenvalue schemes, one for even and one for odd eigenfunctions, is not uncommon (see examples in [20]). The value $N_p=2$ (with $\nu_0=0$ and $\nu_1=1/2$, respectively) is the minimum value allowing for this configuration. The schemes effectively are selected by assigning ν_n values, which are, as in [20], related to possible values of the Frobenius exponent at infinity [see Eqs. (C8) and (154)].

(ii) If N_p grows large, then the ν_n values become densely spaced on the interval $[0,1[$ and, given a λ value that causes congruence, Eq. (154) can be satisfied to any degree of accuracy desired. This would allow one to validate all such λ's as physical eigenvalues. It is an open question whether N_p, in its dependence on the FPE parameters, must stay bounded or is indeed allowed to grow ''sufficiently'' large.

(iii) Eigenfunctions corresponding to eigenvalues from a $\nu_n\neq 0$ scheme are of a different type. They are entirely computable in terms of forward and backward ICF's without a need for recurrent calculation as in Eq. (115). It is conjectured that they are related to the so-called path multiplicative solutions of Heun's equation [8]. The parameter ν_n (which in fact is only determined up to an integer) seems to fulfill the role of ''path index'' (or ''Floquet exponent'').

(iv) Likewise, it is conjectured that the enhanced scheme might be the appropriate one for dealing with spectral accumulation points for the discrete eigenvalues (see the example in [5] mentioned in Sec. I).

C. Continuation by symmetry

The reflection symmetry (5) allows one to define additional representations for the eigenfunctions. The present solution method produces the basic power series (92), which essentially is a Frobenius expansion about the singularity at $e^x=0$:

$$\varphi_k(x)=N_k e^{q_k x}\psi(q_k,-\lambda_k;x)$$
$$=N_k e^{q_k x}\sum_{m=0}^{\infty} a_m(q_k)(-e^x)^m, \tag{156}$$

where the a_m are given by Eq. (93) for the minimal solutions of the standard case, and by the general formula:

$$a_m=\left[T_{(k)}\prod_{j=0}^{m\text{-}1}\frac{B_j}{A_{j+1}}\right]_{z=q_k,p=-\lambda_k} \tag{157}$$

in other cases. The $T_{(k)}$ follow from Eq. (29) or (115), or alternatively, via the $U_{(k)}$ [see, e.g., Eqs. (46), (115), and (133) and equivalents]. An expansion about the singularity at $e^x=\infty$ is directly obtainable by applying the symmetry (5) to Eq. (156). This gives

$$\overline{\varphi_k}(x)=\overline{N_k}e^{-\overline{q_k}x}\sum_{m=0}^{\infty}\overline{a_m}(\overline{q_k})(-e^{-x})^m, \tag{158}$$

where all overbar quantities follow from the original ones by application of the transformation (B5). One notes that $\overline{\lambda_k}=\lambda_k$ and $\overline{q_k}$ is given by the appropriate branch of Eq. (B7):

$$\overline{q_k}=\delta+[\delta^2+\alpha^2(q_k^2+2\mu q_k)]^{1/2}=\delta+[\delta^2-\alpha\lambda_k]^{1/2}. \tag{159}$$

It is easily verified that

$$\overline{A_k}(z,p)=\overline{A_0}(z+k,p)=\alpha^{-1}C_0(-z-k,p)$$
$$=\alpha^{-1}C_{-k}(-z,p),$$
$$\overline{B_k}(z,p)=\alpha^{-1}B_{-k}(-z,p),$$
$$\overline{C_k}(z,p)=\alpha^{-1}A_{-k}(-z,p),$$
$$\overline{D_k}(z,p)=\alpha^{-2}D_{-k-1}(-z,p),$$
$$\overline{Q_k}(z,p)=\alpha^{-1}P^*_{-k}(-z,p),$$
$$\overline{P_k}(z,p)=\alpha^{-1}Q^*_{-k}(-z,p), \tag{160}$$

which completely define $\overline{\varphi_k}(x)$. By extended convergence [8], Eq. (158) is valid for $|e^x|>|\rho^+|$. In their common domain of convergence

$$|\rho^+|<|e^x|<|\rho^-|, \tag{161}$$

φ_k and $\overline{\varphi_k}$ are equal, while outside this ring they are analytical continuations of each other. Analytical continuation of Eq. (158) by Euler's transformation gives the overbar version of Eqs. (94) and (95), useful for the entire physical range $x\in[-\infty,+\infty]$:

$$\overline{\varphi_k}(x)=\overline{N_k}e^{-\overline{q_k}x}(1-\rho^-e^{-x})^{-\overline{\sigma}}\sum_{m=0}^{\infty}\overline{b_m}(\overline{q_k})\overline{X^m},$$

$$\overline{X(x)}=\frac{e^{-x}}{e^{-x}-\overline{\rho^+}}=\frac{\rho^-}{\rho^--e^x}. \tag{162}$$

It must be remarked that the power series in e^{-x} (158) would emerge as a direct result if the operator equation (24) would have been rearranged and factorized as

$$\left(1+\frac{B_0}{C_0}\Delta^{-1}+\frac{A_0}{C_0}\Delta^{-2}\right)\Delta^2\overline{\eta}$$
$$=\left(1+\frac{Q_0^*}{C_0}\Delta^{-1}\right)\left(1+\frac{P_0^*}{C_0}\Delta^{-1}\right)\Delta^2\overline{\eta}=\frac{e^{zx_0}}{C_0} \tag{163}$$

and then solved for $\Delta^2\overline{\eta}$.

A third alternative is solving

$$\left(1+\frac{C_0}{Q_0}\Delta\right)\left(1+\frac{A_0}{Q_0}\Delta^{-1}\right)\Delta\overline{\eta}=\frac{e^{zx_0}}{Q_0} \tag{164}$$

for $\Delta\overline{\eta}$, which generates a Laurent series in e^x,

$$\Delta\overline{\eta}=\frac{e^{zx_0}}{Q_0-Q_0^*}\tilde{\psi}(z,p;x_0),$$

$$\tilde{\psi}(z,p;x)=\sum_{m=0}^{\infty}(-e^x)^m\left(\prod_{j=0}^{m\text{-}1}\frac{C_j}{Q_{j+1}}\right)$$
$$+\sum_{m=1}^{\infty}(-e^{-x})^m\left(\prod_{j=0}^{m\text{-}1}\frac{A_{-j}}{P^*_{-(j+1)}}\right), \tag{165}$$

after a straightforward but rather tedious calculation, using tools from Sec. IV A and Eq. (146). The corresponding eigenfunction representation

$$\tilde{\varphi}_k(x)=\tilde{N}_k e^{q_k x}\tilde{\psi}(q_k,-\lambda_k;x) \tag{166}$$

converges in the ring (161) and is particularly useful for deriving the limiting form in the double confluent case [8] where simultaneously $|\rho^+|\to 0$ and $|\rho^-|\to\infty$ in such a way that the merged singularities at $e^x=0$ and $e^x=\infty$ become irregular (see the example in Sec. V D).

V. CASE STUDIES

Relative to the numerous cases included in the present class of FPE's, the number of case studies discussed here is by necessity very limited. Rather than fully elaborating some general numerical examples (which is almost a straightforward exercise), a brief analytical overview of a few selected cases has been preferred. One reason for this option is that many physically important models happen to be special or limiting cases (most often confluent, double confluent, biconfluent, or triconfluent) of Heun's equation. The limiting procedures necessary to extract such a particular case result from the general solution are not always obvious and often have an *ad hoc* character. The selected examples will contain some additional analytical material related to this problem. Another reason is that the present study generalizes the ''unifying stochastic process'' FPE (see Sec. I and [1]). It is instructive to verify the ''downward'' compatibility of the actual results by retrieval of the three hypergeometric limit cases that are apparent from Eq. (1).

A. First hypergeometric case

A first hypergeometric case is obtained parametrically by setting $\beta=\gamma=0$ in Eq. (2). The resulting FPE, devoid of first powers of e^x, is identical in structure to the unifying FPE of [1] [except for an inessential scale factor 2 in x, which can be removed by a trivial reparametrization with $\rho=1/2$ in Eqs. (3) and (4)]. With $\beta=0$, condition (42) for ICF convergence is violated, but the factorizers P_0 and Q_0 remain defined by Eq. (A21) (with a nonminimal M), while Eq. (A22) with $B_0\equiv 0$ basically reduces to a solvable first-order difference equation for M. One has

$$Q_0(z,p)=-P_0(z,p)=\frac{M_0(z,p)}{M_1}=-C_0A_1\frac{M_2}{M_1},$$

$$M(z,p)=\frac{(e^{i\pi}/16\alpha)^{z/2}}{\Gamma\left(\frac{z-c^+}{2}\right)\Gamma\left(\frac{z-c^-}{2}\right)\Gamma\left(\frac{z-a^++1}{2}\right)\Gamma\left(\frac{z-a^-+1}{2}\right)}, \tag{167}$$

with $a^\pm(p),c^\pm(p)$ from Eq. (B8). The physical branch equation for the q_k values can be identified as

$$q_k-c^-(\alpha p_0(q_k))=q_k+\delta+[\delta^2+\alpha(q_k^2+2\mu q_k)]^{1/2}=-2k,$$
$$k=0,1,..., \tag{168}$$

and the hypergeometric results of [1] can be directly reconstructed. This example clearly illustrates that knowledge of the M function becomes essential whenever the ICF's do not exist. The above hypergeometric case result may be taken as a zeroth-order approximation for M in the analysis of close-to-hypergeometric cases with small but nonzero β and γ, where the ICF's do not exist. Finally, it should be mentioned that the present case is not documented as a known hypergeometric limit of Heun's equation [8,19]. It may be extracted from Eq. (C6) by taking the fourth singularity $a=-1$, the accessory parameter $b=0$, and a quadratic transformation $y'=y^2$ of the independent variable.

B. Second and third hypergeometric cases

The second hypergeometric case corresponds to Eq. (2) without e^{2x} terms. Their annihilation has to proceed via the fully parametrized FPE (1) by zeroing a, b, and h in a consistent way. Eventually, the following procedure results. Replace

$$x\to x+\ln\omega,\quad \omega>0$$
$$\beta\to\beta/\omega,\quad \gamma\to\gamma/\omega \tag{169}$$

in Eq. (2) and in all quantities of the general case solution. By letting now $\omega\to 0$, Eq. (2) acquires the structure of the unifying FPE, while the solution components behave as follows. The convergence condition (42) for the ICF's obviously is satisfied for sufficiently small ω and Q_0 is seen to behave asymptotically as

$$Q_0(z,p)=\frac{M_0}{M_1}\sim\frac{1}{\omega}B_0(z,p), \tag{170}$$

with B_0 as originally in Eq. (22), such that the M function ''degenerates'' to [see Eq. (B8) for $b^\pm(p)$]

$$M(z,p)=M_0=\frac{(-\omega/\beta)^z}{\Gamma(z-b^+)\Gamma(z-b^-)}. \tag{171}$$

For a stochastically stable limit process ($\epsilon<0$) the zeros of the physical branch of $M(q,\alpha p_0(q))=0$ are given by

$$q_k-b^-(\alpha p_0(q_k))=q_k+\epsilon+\left(\epsilon^2+\frac{\alpha\gamma}{\beta}(q_k^2+2\mu q_k)\right)^{1/2}$$
$$=-k,\quad k=0,1,..., \tag{172}$$

which, as in [1], can be explicitly solved for the q_k. Eigenfunctions are retrieved most simply by starting from the general expression (27) for ψ, in which e^x and B_j are replaced by ωe^x and B_j/ω, respectively. One gets

$$\lim_{\omega\to 0}\psi(z,p;x)=\sum_{m=0}^{\infty}(-e^x)^m\left(\prod_{j=0}^{m-1}\frac{B_j}{A_{j+1}}\right), \tag{173}$$

which, when inserted into Eq. (74), nicely truncates to the expected polynomials of [1]. The limiting normalization $\lim_{\omega\to 0}(N_k\omega^{q_k})$ will not be calculated here. Referring to [19], the above case corresponds to the hypergeometric limit of Heun's equation for $a\to\infty$, $b\to\infty$, and $b/a=c$.

The third hypergeometric case, with all e^x-independent terms in Eq. (2) suppressed, is equivalent to a second case in terms of $y=-x$ and parameters $1/\alpha$, β/α, γ/α, $-\mu$, $-\epsilon$, $-\delta$, by virtue of the reflection symmetry (5).

C. More hypergeometrics

There are many more instances in which Heun's equation (C6) reduces to the Gauss hypergeometric equation [17]. Only the most obvious ones, following by mere inspection of the equation, have been listed in [8]. Reference [19] mentions two nontrivial cases where a transformation of variables together with particular parameters perform the reduction (the case in Sec. V A above is of a similar type). It can be shown that whenever a singular point of Eq. (C6), say, the point a, merges with one of the others $(0,1,\infty)$ while keeping the exponents $(A,B,\ldots)$ finite, an equation results that generally is not identical but reducible to the Gauss hypergeometric equation by a simple transformation of the type $F^* = z^p F$ or $F^* = (z-1)^p F$. The identification of possible subclasses of the FPE (2) corresponding to these configurations is easy and will not be pursued here. The marginal case $\beta^2 = 4\alpha$ (or $\rho^+ = \rho^- = -\beta/2$), where the ICF's cease to exist and with $\delta + \mu = 2\epsilon$ to keep $\nu^\pm$ finite, is a noteworthy example.

D. Fully discrete spectra

The analytical results of Sec. III and the extensions in Sec. IV have been fully elaborated for the discrete part of the spectrum only. Cases with entirely discrete spectra become particularly interesting as their analysis is completely covered by the presently available material. A fully discrete spectrum is obtained when the equivalent Schrödinger potential (18) becomes confining, i.e., when the asymptotic energy levels $\alpha\mu^2$ and δ^2/α [Eq. (19)] grow to infinity. The corresponding limiting procedure must be such that the FPE resulting from Eq. (2) stays meaningful, and that the spectral representation in at least one of the q or $\overline{q}$ planes survives with an infinite discrete spectrum locus (intersection and branch points being moved to infinity). In a first example, the option is

$$\mu \to \infty, \quad \alpha = \frac{\alpha^*}{\mu}, \quad \alpha^*, \delta \text{ finite.} \tag{174}$$

The FPE reduces to

$$\frac{\partial w(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}\left[\frac{(e^x+\beta)e^x}{\gamma e^x+1} w\right] - 2\frac{\partial}{\partial x}\left[\frac{\delta e^{2x}+\beta\epsilon e^x+\alpha^*}{\gamma e^x+1} w\right], \tag{175}$$

for which obviously the reflection symmetry is broken. Equation (175) actually defines a four-parameter subclass of Eq. (2). One parameter in Eq. (175) is redundant, but its elimination by translation of x, scaling of t, and reparametrization is not essential for the subsequent discussion. The limiting form of the steady-state PDF (80) is easily obtained

$$w_S(x) = \tilde{N}_S (1+\gamma e^x)(1+\beta e^{-x})^{\nu^- -1} \times \exp\left[2(\delta-1)x - \frac{2\alpha^*}{\beta} e^{-x}\right], \tag{176}$$

with ν^- from Eq. (83):

$$\nu^- = 2\left[\frac{\alpha^*}{\beta^2} + \delta - \epsilon\right]. \tag{177}$$

In the q plane, the branch points (64)

$$q_C^\pm = \mu\left\{-1 \pm \left[1 - \left(\frac{\delta}{\alpha^*}\right)^2\right]^{1/2}\right\} \tag{178}$$

and the locus intersection $q = -\mu$ are seen to move to infinity with μ, irrespective of whether the $q_C^\pm$ are real or complex. Because the convergence condition (42) is always satisfied, the complete subclass (175) is solvable in terms of the ICF's, which have a well-defined finite limiting form. Some of the components simplify, e.g.,

$$\lambda(q) = \lim_{\mu\to\infty} [-\alpha p_0(q)] = -2\alpha^* q,$$

$$A_0(z,p) = p - 2\alpha^* z, \tag{179}$$

$$C_0(z,p) = -(z^2 + 2\delta z).$$

In the limit $\rho^+ = 0$ and $\rho^- = -\beta$, such that Eq. (175) can be related to the (simply) confluent Heun equation [8]. For the eigenfunctions $\varphi_k(x)$, an appropriate form must be chosen that survives the limiting operations. A first candidate is the continuation (94) and (95) with ρ^+ replaced by $\rho^- = -\beta$ (see remark preceding those equations):

$$\varphi_k(x) = N_k e^{q_k x}\left(1 + \frac{e^x}{\beta}\right)^{-\sigma} \sum_{m=0}^{\infty} b_m(q_k)\left(\frac{e^x}{e^x+\beta}\right)^m, \tag{180}$$

with the b_m according to Eq. (95) (with also $\rho^+ \to \rho^- = -\beta$). The a_m figuring in Eq. (95) are given by the limiting form of Eqs. (93) or (157) according to whether the eigenvalues result from a standard, a nonstandard, or a congruent search. Alternatively, the limiting form of continuation (162) can be used, where it should be noted that [see Eq. (159)] $\lim_{\mu\to\infty} \overline{q}_k = 0$. (In the limit, the broken reflection symmetry invalidates the analysis in the $\overline{q}$ plane, where the intersection $\overline{q} = \delta$ and the branch points remain unchanged. This suggests that for large but finite μ and small but nonzero α, an accumulation of discrete eigenvalues near $\overline{q} = 0$ takes place in the $\overline{q}$ plane.) The present subclass has only three free zeros

$$a^+(p) - 1 = \frac{p}{2\alpha^*} - 1, \quad c^+(p) = -2\delta, \quad c^-(p) = 0 \tag{181}$$

and the candidate c^--congruent q values have the simple expression [from Eq. (113)]

$$q(N_c) = 1 - N_c \tag{182}$$

such that condition (114) reduces to

$$Q_0(1, -\lambda) = 0, \quad \lambda = 2\alpha^*(N_c - 1). \tag{183}$$

It is an open question whether c^+ congruence should be considered too in this limiting case, as there are strictly no symmetry arguments anymore to forbid this (see, however, the following subclass). A further detailed treatment including the relation to the confluent Heun equation, the identification of confluent hypergeometric cases, and other topics clearly is a vast subject in its own that falls outside the scope of this preliminary case study.

A second discrete spectrum subclass is obtained from Eq. (2) by

$$\alpha\to\infty, \quad \delta=\alpha\delta^*, \quad \delta^*,\mu \text{ finite}. \tag{184}$$

To yield again a meaningful FPE with a maximal number of parameters (four plus one redundant parameter) and suitable for ICF treatment, one can set also

$$\beta=\alpha\beta^*, \quad \gamma=\alpha\gamma^* \tag{185}$$

such that the limiting equation becomes

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial^2}{\partial x^2}\left[\frac{\beta^* e^x+1}{(e^x+\gamma^*)e^x}w\right] -2\frac{\partial}{\partial x}\left[\frac{\delta^* e^{2x}+\beta^*\epsilon e^x+\mu}{(e^x+\gamma^*)e^x}w\right]. \tag{186}$$

This equation exactly has the structure of the reflected $(x\to -x)$ version of Eq. (175) and hence needs no separate discussion. When identity with this version is achieved by a suitable choice of the parameters, Eqs. (175) and (186) define two problems that are mutually conjugate under reflection and hence isospectral. This property can be used to develop adapted criteria for physicality of the eigenvalues in both problems.

For a third and essentially different discrete spectrum subclass, confinement is attained via

$$\omega\to\infty, \quad \mu=\omega\mu^*, \quad \delta=\omega\delta^*, \quad \alpha,\mu^*,\delta^* \text{ finite}. \tag{187}$$

To keep a meaningful diffusion equation, one may additionally scale β and the time variable t,

$$\beta=\omega\beta^*, \quad t=\frac{t^*}{\omega}, \tag{188}$$

such that the limit of Eq. (2) becomes (dropping the asterisks)

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial^2}{\partial x^2}\left(\frac{\beta e^x}{\alpha e^{2x}+\gamma e^x+1}w\right) -2\frac{\partial}{\partial x}\left(\frac{\delta e^{2x}+\beta\epsilon e^x+\alpha\mu}{\alpha e^{2x}+\gamma e^x+1}w\right). \tag{189}$$

The equation has two redundant parameters and the reflection symmetry (5) is conserved. The steady-state PDF (80) reduces to

$$w_S(x)=\tilde{N}_S(\alpha e^x+\gamma+e^{-x})\,\exp[2\epsilon x+2\beta^{-1}(\delta e^x-\alpha e^{-x})] \tag{190}$$

and the other solution ingredients behave as follows for large ω.

(i) The singularities $\rho^\pm$ [Eq. (81)] simultaneously move

$$\rho^+\sim-\frac{\alpha}{\beta}\frac{1}{\omega}\to 0, \quad \rho^-\sim-\beta\omega\to\infty \tag{191}$$

such that the present class can be related to Heun's double confluent equation [8], and the Laurent series (165) gives the appropriate representation for deriving the limiting form of the eigenfunctions.

(ii) Because $\beta\to\infty$, the ICF's stay convergent and their asymptotics for large ω can be derived. A sensitive point hereby is that the actual arguments have to be substituted [e.g., as in $Q_0(q,\alpha p_0(q))$ or in $Q_0(d_i(-\lambda),-\lambda)$] before the asymptotics are determined because these arguments may depend on ω as well. Starting with the eigenvalue expression

$$\lambda(q)=-\alpha p_0(q)\sim-2\alpha\mu q\omega=\tilde{\lambda}(q)\omega, \tag{192}$$

it is seen that λt stays finite, as it should (asterisks removed in the right equality):

$$\lambda t=-\alpha p_0(q)t\sim-2\alpha\mu q\omega\left(\frac{t}{\omega}\right)=\tilde{\lambda}t. \tag{193}$$

Hereby, it is tacitly assumed that the q values, if ω dependent at all, stay bounded when $\omega\to\infty$. The elements of the ICF's Q_0 and Q_0^*, with arguments as in the most general eigenvalue search procedure (153), behave as

$$A_k(q+\nu_n,-\lambda(q))\sim[-\tilde{\lambda}(q)-2\alpha\mu(q+k+\nu_n)]\omega=\tilde{A}_k\omega,$$

$$B_k(q+\nu_n,-\lambda(q))\sim\{-\gamma\tilde{\lambda}(q)-\beta[(q+k+\nu_n)^2+2\epsilon(q+k+\nu_n)]\}\omega=\tilde{B}_k\omega,$$

$$C_k(q+\nu_n,-\lambda(q))\sim[-\alpha\tilde{\lambda}(q)-2\delta(q+k+\nu_n)]\omega=\tilde{C}_k\omega,$$

$$\tilde{\lambda}(q)=-2\alpha\mu q \tag{194}$$

such that for Q_n, e.g., it follows that

$$Q_n\sim\tilde{Q}_n\omega, \tag{195}$$

where $\tilde{Q}_n$ is an ω-independent ICF in terms of $\tilde{A}_k,\tilde{B}_k,\tilde{C}_k$:

$$\tilde{Q}_n=\tilde{B}_n-\frac{\tilde{C}_n\tilde{A}_{n+1}|}{|\tilde{B}_{n+1}}-\frac{\tilde{C}_{n+1}\tilde{A}_{n+2}|}{|\tilde{B}_{n+2}}-\cdots. \tag{196}$$

Note that $\tilde{A}_k$ is q independent, while $\tilde{C}_k$ is linear in q. It is assumed that ν_n can be assigned independently of ω or at least stays bounded when $\omega\to\infty$. In this case the q values obtained from $\tilde{Q}_n-\tilde{Q}_n^*=0$ [as a limiting version of Eq. (153)] will be ω independent indeed.

(iii) The free zeros behave for large ω as

$$a^+(-\lambda)-1\sim-\frac{\lambda}{2\alpha\mu}\frac{1}{\omega}-1=-\left(1+\frac{\tilde{\lambda}}{2\alpha\mu}\right)=q-1,$$

$$a^-(-\lambda)-1\sim-2\mu\omega,$$

$$c^+(-\lambda)\sim-2\delta\omega,$$

$$c^-(-\lambda)\sim-\frac{\alpha\lambda}{2\delta}\frac{1}{\omega}=-\frac{\alpha\tilde{\lambda}}{2\delta}=\frac{\alpha^2\mu q}{\delta} \tag{197}$$

and only the two useful zeros $a^{+}-1$ and c^{-} are seen to survive. Again, candidate c^{-}-congruent q values get a simple ω-independent expression [e.g., from Eq. (154)]

$$q(N_c+\nu_n)=\frac{N_c-1+\nu_n}{\dfrac{\alpha^2\mu}{\delta}-1} \tag{198}$$

and congruence detection [see Eq. (155)] can be expressed in terms of the asymptotic ICF's such as, e.g., Eq. (195).

A final remarkable discrete spectrum example is obtained from the same confinement principle (187), but instead of time scaling as in Eq. (188), the growth of β, δ, and μ is compensated by the growth of γ. The overall procedure can be tuned so as to avoid redundancies

$$\gamma\to\infty,\quad \delta=-\sigma\gamma,\quad \mu=\sigma\gamma,\quad \alpha=1,\quad \beta=\gamma,\quad \sigma>0, \tag{199}$$

yielding the equation

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial^2 w}{\partial x^2}-2\frac{\partial}{\partial x}[(\epsilon-2\sigma\sinh x)w]. \tag{200}$$

This is the FPE for the hyperbolic sine model, whose analysis in [3,20] significantly inspired the present research. A particularity for this limiting case is the necessity to scale the q values during the limiting process. As time does not scale, the eigenvalues λ must stay finite. They are given by

$$\lambda(q)=-(q^2+2\sigma\gamma q) \tag{201}$$

such that the scaling law for q is

$$q=\frac{\tilde{q}}{\gamma},\quad \tilde{q}=-\frac{\lambda}{2\sigma}. \tag{202}$$

Because $\alpha=1$ and $\mu=-\delta$, the q and $\overline{q}$ planes are identical and both show accumulation of q values at zero (to be observed for large but finite γ). In the limit, the analysis survives, however, in the $\tilde{q}$ plane. The following asymptotics for large γ are easily obtained

$$\rho^{+}\sim-\gamma^{-1},\quad \rho^{-}\sim-\gamma,$$

$$a^{+}(-\lambda)\sim\tilde{q}\gamma^{-1},\quad a^{-}\sim-2\sigma\gamma,$$

$$c^{+}\sim2\sigma\gamma,\quad c^{-}(-\lambda)\sim-\tilde{q}\gamma^{-1}, \tag{203}$$

showing a double confluence and yielding the asymptotic congruence condition (154)

$$c^{-}(-\lambda)-q\sim-\frac{2\tilde{q}}{\gamma}=N_c-1+\nu_n, \tag{204}$$

which is identically satisfied for all congruence generating $\tilde{q}$ values in the limit, with $N_c=1$ and $\nu_n=0$. The ICF building blocks behave as

$$A_k(q+\nu_n,-\lambda(q))\sim-2\sigma(k+\nu_n)\gamma=\tilde{A}_k\gamma,$$

$$B_k(q+\nu_n,-\lambda(q))\sim[-\lambda-(k+\nu_n)^2-2\epsilon(k+\nu_n)]\gamma=\tilde{B}_k\gamma,$$

$$C_k(q+\nu_n,-\lambda(q))\sim2\sigma(k+\nu_n)\gamma=\tilde{C}_k\gamma \tag{205}$$

such that the asymptotics of the ICF's are similar to Eqs. (195) and (196). The ICF's for congruence detection (147) and (148) can be constructed similarly from the expressions (205), by formally setting $\nu_n=1$ and $\nu_n=0$, respectively. Finally, the steady-state PDF becomes

$$w_S(x)=\tilde{N}_S\exp[2\epsilon x-4\sigma\cosh x], \tag{206}$$

where $\tilde{N}_S$ is expressible in terms of a modified Bessel function [3].

VI. SUMMARY AND CONCLUSIONS

The one-dimensional FPE (2) studied in this paper generalizes the FPE for a unifying stochastic Markov process [1] by allowing for nonmonotonic drift and diffusion coefficients. As a consequence, the equation not only unifies a large number of known stochastic processes from different branches of physics, which previously had been solved independently, but it also covers a wide variety of different, more general stochastic systems. These systems are characterized by a more complex (e.g., saturating) state dependence of the stochastic forces determining the process, and a subclass of them even exhibits bimodality of its equilibrium distribution. The equation basically introduces the general, i.e., the nonconfluent, Heun equation into the domain of diffusion theory. By having one more finite singular point, it generalizes the Gauss hypergeometric equation, to which it reduces in several instances.

The constructive solution method [6,1] has been extended successfully for the present problem by factorization of the Laplace-Fourier transformed FPE operator. The necessary theory about factorizers has been developed in Appendix A, with the entire function representation and the principles of congruence as main results.

Under the restriction $\beta^2>4\alpha$, which ensures the existence of ICF factorizers, fully analytical results have been obtained, except for the eigenvalues. These have to be found from ICF equations that allow for numerical access to the zeros of a generally unknown spectral kernel function. It has been shown that if the most general factorizers are used, then eventually both forward and backward ICF's are invoked to define the solution structure and the eigenvalue scheme. Occasionally, the general solution may reduce to the standard case, which, for tutorial reasons, has been fully elaborated first and where forward ICF's alone can do the job.

The present solution method does not explicitly refer to Heun's equation for its development. It has been thought useful, however, to identify the results obtained with the different known types of solutions that are listed in Ref. [8]. Hereby, it appears that the present study not only unifies a significant number of physically important stochastic processes but also achieves a unification of the distinct types of Heun's equation eigenvalue problems. The important role of congruence should be clear in this respect, as considerably enlarging the scope of continued fraction solution methods.

The preliminary results in the case studies indicate that also solutions for at least some of the confluent versions of Heun's equation are obtainable by appropriate limiting processes upon the general solution. As already observed in [1], this embedding procedure stays possible even if the limiting FPE itself cannot be solved anymore by the present method. At this stage it should be clear that the solution method also applies to stochastically unstable cases where of course the normalization has to be reconsidered. Without claiming completeness, it may safely be concluded that the FPE (2) can be solved in all cases of physical interest. Finally, a short list of possible future research topics can be given: other singularity configurations, such as $\rho^{\pm}$ complex conjugate (for $\beta^2<4\alpha$ and the ICF's nonexisting), or ρ^+ and/or ρ^- real and positive and thus defining finite e^x-interval problems for the same equation; analytical prediction of the number of eigenvalues from the FPE parameter set $\{\alpha,\beta,\gamma,\delta,\epsilon,\mu\}$; identification of other physically relevant special or limiting cases (e.g., with bimodal steady-state PDF's); adaptation of the actual solution method to generate the known [8] hypergeometric function series solutions for Heun's equation, which may have superior convergence properties; and derivation of additional properties of ICF's, considered as special factorizers.

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APPENDIX A: FACTORIZERS

The factorization of the operator (39) supposes the existence of a pair of functions $R_0(z,p)$ and $S_0(z,p)$ solving the set of nonlinear first-order FRE's (or difference equations)

$$R_0+S_0=B_0,$$

$$R_0S_1=D_0=C_0A_1. \tag{A1}$$

Here A_0,B_0,C_0 are given polynomials in z and p [see Eq. (22)] and $F_k(z)=\Delta^kF_0(z)=F_0(z+k)$ as usual. For convenience, the fourth-order polynomial D_0 will be denoted as

$$D_0(z,p)=\alpha(z-d_1)(z-d_2)(z-d_3)(z-d_4), \tag{A2}$$

where $\{d_1,d_2,d_3,d_4\}$ stands for any permutation of the zeros of C_0 and A_1, i.e., $c^{\pm}(p)$ and $a^{\pm}(p)-1$ [see Appendix B, Eq. (B8)]. The p dependence of R_0 and S_0 exclusively comes in via these $d_i(p)$ and via $b^{\pm}(p)$. It will be suppressed in the notation whenever it is less essential. The system (A1) can be split up in two separate equations for R_0,S_0:

$$R_0=\frac{D_0}{B_1-R_1},\quad S_0=B_0-\frac{D_0}{S_1}. \tag{A3}$$

1. General properties

It is assumed (in fact, suggested by the structure of the directly emerging infinite continued fraction solutions; see Sec. III and Sec. 3 below) that solutions of Eq. (A1) are meromorphic functions of z, having in general an infinite number of poles and zeros. From Eq. (A1) it appears that R_0 and S_0 must have the same poles with opposite principal parts (residues, if the poles are simple) because B_0 is entire. Poles of R_0 must be zeros of S_1 and poles of S_1 [i.e., of $S_0(z+1)$] must be zeros of R_0 because D_0 is entire. In addition to these mutually compensating poles and zeros, R_0 and S_1 must have the free zeros of D_0 and the constant α in some way distributed between them. Without loss of generality $d_1,d_2,f\alpha^{1/2}$ can be assigned to R_0 and $d_3,d_4,f^{-1}\alpha^{1/2}$ to S_1, with f an arbitrary constant. Let now $\{\pi_k(p)\}$ be the set of poles of R_0 and S_0. By well-known theorems of complex function theory [12,13], there exists an entire function $N(z,p)$ that has exactly all the zeros $z=\pi_k$ with correct multiplicity and no others. If $\{\pi_k\}$ were known, then N could be constructed, e.g., as an infinite product of monomials $(1-z/\pi_k)$ and appropriate but not uniquely determined convergence factors (Hadamard's theorem). Now $R_0N(z)$ is free of poles and $R_0/N(z+1)$ has only the assigned free zeros d_1 and d_2. It follows that

$$R_0\frac{N_0}{N_1}=E_0(z)e^{k(z)},\quad E_0=-f\alpha^{1/2}(z-d_1)(z-d_2), \tag{A4}$$

where $k(z)$ is entire and thus e^k is entire without zeros. Similarly, for S_0 and with $h(z)$ entire,

$$S_0\frac{N_0}{N_{-1}}=F_{-1}(z)e^{h(z)},\quad F_0=-f^{-1}\alpha^{1/2}(z-d_3)(z-d_4). \tag{A5}$$

Note that $E_0F_0=D_0$. Substitution of Eqs. (A4) and (A5) into Eq. (A1) yields

$$k(z)=-h(z+1),$$

$$E_0e^{k(z)}\frac{N_1}{N_0}+F_{-1}e^{-k(z-1)}\frac{N_{-1}}{N_0}=B_0. \tag{A6}$$

A simple asymptotic analysis shows that $k(z)$ must be constant (say, zero) while $N_0(z)$ behaves asymptotically as

$$N_0(z)\sim r^zz^{\omega}\left[1+O\left(\frac{1}{z}\right)\right], \tag{A7}$$

with [see Eq. (81) for $\rho^{\pm}$]

$$r=r^{\pm}=f^{-1}\left(\frac{\rho^{\pm}}{\rho^{\mp}}\right)^{1/2},$$

$$\omega=\omega^{\pm}=\frac{\rho^{\pm}(d_1+d_2)+\rho^{\mp}(d_3+d_4+2)-2\beta\epsilon}{\rho^{\pm}-\rho^{\mp}}. \tag{A8}$$

It is convenient to define

$$K_0(z,p)=N_{-1}(z,p)=N_0(z-1,p). \tag{A9}$$

K_0 is an entire function having now the set of zeros $\{\rho_k(p) = \pi_k(p)+1\}$ (i.e., the nonfree zeros of S_0) and the important result follows that the factorizers can be expressed as

$$R_0(z,p)=E_0(z,p)\frac{K_2(z,p)}{K_1(z,p)},$$

$$S_0(z,p)=F_{-1}(z,p)\frac{K_0(z,p)}{K_1(z,p)}, \qquad \text{(A10)}$$

where K_0 satisfies the FRE

$$E_0K_2-B_0K_1+F_{-1}K_0=0 \qquad \text{(A11)}$$

and is determined up to a periodic in z.

2. Congruent zeros: Zero movers

Zeros common to, e.g., K_0 and K_1 will, according to Eq. (A10), not show up in the factorizer S_0. Otherwise, Eq. (A3) seems to forbid a common zero for S_0 and S_1, unless when coinciding with a zero d_i of D_0. This suggests that common zeros, if possible at all, deserve special attention. Therefore, let $(a,a-1)$ be a pair of unitary spaced zeros of K_0:

$$K_0=(z-a)(z-a+1)\overline{K_0}. \qquad \text{(A12)}$$

From Eq. (A10) one finds

$$R_0=E_0\frac{(z+3-a)}{(z+1-a)}\frac{\overline{K_2}}{\overline{K_1}}, \quad S_0=F_{-1}\frac{(z-a)}{(z+2-a)}\frac{\overline{K_0}}{\overline{K_1}}, \qquad \text{(A13)}$$

which violates the equal poles demand of Eq. (A1) unless both poles are compensated by other zeros. Four possibilities can be distinguished.

(i) For compensation by $\overline{K_0}$ and $\overline{K_2}$, K_0 then additionally has the zeros $a+1$ and $a-2$, and the new pairs $(a+1,a)$ and $(a-1,a-2)$ need compensation again. If this way of compensation prevails, then K_0 gets a double infinite series of unitary spaced zeros $a+k, k=-\infty,\ldots,-1,0,1,\ldots,\infty$. The value of a can be arbitrary, but a periodic factor having only these zeros can be divided out from K_0.

(ii) For compensation by E_0 and $\overline{K_0}$, $a=d_i+1$ ($i=1$ or 2) and K_0 additionally has the zero $a-2$. Continued compensation by $\overline{K_0}$ generates for K_0 a zero series $a-k, k=0,1,2,...,\infty$, left congruent to d_i. A left-congruent K_0 can be expressed as

$$K_0=\frac{b^{-z}\widetilde{K}_0}{\Gamma(z-d_i-1)}, \qquad \text{(A14)}$$

where $\widetilde{K}_0$ is entire and noncongruent with respect to d_i and solves the equation

$$\frac{E_0}{b(z-d_i)}\widetilde{K}_2-B_0\widetilde{K}_1+b(z-d_i-1)F_{-1}\widetilde{K}_0=0, \qquad \text{(A15)}$$

which displays a distribution of free zeros and of α different from the one in Eq. (A11); d_i has been moved from E_0 to F_0 and the arbitrary constant b redistributes the constant α. Left-congruent zero series do not show up in the factorizers, except for the first zero d_i+1 as a free zero of S_0.

(iii) The scheme of compensation by $\overline{K_2}$ and F_{-1} similarly results in a right-congruent zero series, starting at d_j+2 ($j=3$ or 4) and a corresponding zero move from F_0 to E_0:

$$K_0=\frac{(-c)^z\hat{K}_0}{\Gamma(d_j+2-z)},$$

$$c(z-d_j)E_0\hat{K}_2-B_0\hat{K}_1+\frac{F_{-1}}{c(z-d_j-1)}\hat{K}_0=0. \qquad \text{(A16)}$$

If such a series is present, then only the moved zero d_j will show up in R_0.

(iv) The scheme of compensation by E_0 and F_{-1} allows for a finite unitary spaced zero series with end points d_i+1 ($i=1$ or 2) and d_j+2 ($j=3$ or 4), which then must have a nonzero integer spacing. The free zeros d_i and d_j are interchanged when such a series is present in K_0.

It is concluded that factorizers cannot have unitary spaced zeros (except for two free zeros). Their underlying K_0 function can have them, but then in congruent (or doubly infinite) series only. Similarly it can be shown that K_0 cannot have d_i+1 or d_j+2 ($i=1$ or 2 and $j=3$ or 4) as an isolated zero without possessing the whole corresponding congruent series.

If the entire function K_0 has no unitary spaced and hence no congruent series of zeros, then, e.g., Eq. (A14) can nevertheless be considered merely as a transformation of the dependent variable in Eq. (A11) interrelating the solutions of two equations having a distinct free zero distribution. As such, the factors

$$m_i^*(z)=\frac{b^{-z}}{\Gamma(z-d_i-1)}, \quad m_j=\frac{(-c)^z}{\Gamma(d_j+2-z)} \qquad \text{(A17)}$$

figuring in Eqs. (A14) and (A16), respectively, can concisely be termed ''zero movers'' as their application to the dependent variable induces a change of the position of a free zero in the defining difference equation. It should be noted that, in general, solutions of equations like Eqs. (A11), (A15), and (A16) are not entire but only meromorphic. However, because in this case poles always occur in congruent positions, entire solutions of the same equations can always be obtained by multiplying with a periodic function having congruent zeros.

3. Infinite continued fractions

Normally, the determination of any pair of factorizers [e.g., R_0,S_0 in Eq. (A10)] supposes solving the FRE for the underlying entire function K_0 [e.g., Eq. (A11)]. This generally is impossible or at least as difficult as the original problem generating the factorizer equations (A1). There are, however, special directly accessible solutions of Eq. (A1). By forward iteration of Eq. (A3) one finds

$$R_0=P_0(z,p)=\frac{D_0}{B_1-P_1}=\frac{D_0|}{|B_1}-\frac{D_1|}{|B_2}-\cdots\frac{D_k|}{|B_{k+1}}-\cdots,$$

$$S_0=Q_0(z,p)=B_0-P_0=B_0-\frac{D_0|}{|B_1}-\cdots\frac{D_k|}{|B_{k+1}}-\cdots. \qquad \text{(A18)}$$

The bivariate ICF's P_0 and Q_0 are directly computable if they converge, i.e., under the condition [11]

$$\beta^2 > 4\alpha. \tag{A19}$$

Similarly, by backward iteration of Eq. (A3)

$$P_0^* = B_0 - \frac{D_{-1}|}{|B_{-1}} - \cdots \frac{D_{-k}|}{|B_{-k}} - \cdots,$$

$$Q_0^* = \frac{D_{-1}}{P_{-1}^*} = \frac{D_{-1}|}{|B_{-1}} - \cdots \frac{D_{-k}|}{|B_{-k}} - \cdots, \tag{A20}$$

which is another pair of solutions, convergent under the same condition (A19).

As factorizers, the ICF's P_0 and Q_0 have the formal representation

$$P_0 = D_0 \frac{M_2}{M_1}, \quad Q_0 = \frac{M_0}{M_1}, \tag{A21}$$

where D_0 is completely assigned to P_0 as indicated by the structure of the ICF's and M_0 is the (forward) ''minimal'' solution [2,11] of

$$D_0 M_2 - B_0 M_1 + M_0 = 0. \tag{A22}$$

Clearly, the zeros of this minimal solution show up as zeros of the ICF Q_0 provided no congruence occurs. Similarly, for the backward ICF's, one has

$$P_0^* = \frac{M_2^*}{M_1^*}, \quad Q_0^* = D_{-1} \frac{M_0^*}{M_1^*}, \tag{A23}$$

where M_0^* is the backward minimal solution of the equation ''adjoint'' to Eq. (A22):

$$M_2^* - B_0 M_1^* + D_{-1} M_0^* = 0. \tag{A24}$$

From Eqs. (A22) and (A24) it follows that

$$D_0 M_2 M_1^* - M_1 M_2^* = \Delta(D_{-1} M_1 M_0^* - M_0 M_1^*)$$
$$= D_{-1} M_1 M_0^* - M_0 M_1^*, \tag{A25}$$

showing this expression to be periodic (or constant) in z.

4. General factorizers

Any other factorizer pair, e.g., Eq. (A10), can now indirectly be related to the ICF's via the minimal solutions M_0 and M_0^*. The general solution K_0 of Eq. (A11), e.g., can be written as

$$K_0 = \omega(z)\varphi_0(z) M_0 + \omega^*(z)\varphi_0^*(z) M_0^*. \tag{A26}$$

The first term defines a solution of Eq. (A11) by moving the two free zeros d_3, d_4 in Eq. (A22). The second term is obtained by oppositely moving d_1, d_2 in Eq. (A24); ω and ω^* are arbitrary periodic functions or constants and φ_0, φ_0^* are appropriate products of zero movers (A17). It is sufficient to know that they are solutions of

$$\varphi_1 = F_{-1}\varphi_0, \quad \varphi_0^* = E_{-1}\varphi_1^*, \tag{A27}$$

with

$$\frac{\varphi_1}{\varphi_0}\frac{\varphi_0^*}{\varphi_1^*} = D_{-1}. \tag{A28}$$

Constructing now, e.g., the factorizer S_0 [Eq. (A10)] from K_0 [Eq. (A26)], one finds

$$S_0 = F_{-1}\frac{K_0}{K_1} = F_{-1}\frac{\omega\varphi_0 M_0 + \omega^*\varphi_0^* M_0^*}{\omega\varphi_1 M_1 + \omega^*\varphi_1^* M_1^*} \tag{A29}$$

or, upon simplification by use of Eq. (A27),

$$S_0 = \frac{M_0 + \bar{\omega}\dfrac{\varphi_0^*}{\varphi_0} M_0^*}{M_1 + \bar{\omega}\dfrac{\varphi_1^*}{\varphi_1} M_1^*}. \tag{A30}$$

Likewise, for R_0,

$$R_0 = D_0 \frac{M_2 + \bar{\omega}\dfrac{\varphi_2^*}{\varphi_2} M_2^*}{M_1 + \bar{\omega}\dfrac{\varphi_1^*}{\varphi_1} M_1^*}. \tag{A31}$$

In this representation, R_0 and S_0 appear as mere generalizations of the forward ICF's P_0 and Q_0, obtained by replacing the forward minimal M_0 in Eq. (A21) by a general, and thus forward dominant, solution of Eq. (A22). The above factorizers are in fact the general solutions of Eq. (A1) as it is easily verified that by choosing the periodic functions, all possible free zero assignements can be obtained. (Note, e.g., that for $\bar{\omega}=0$, P_0 and Q_0 are retrieved, while for $\bar{\omega}=\infty$, one gets P_0^* and Q_0^*.)

APPENDIX B: BRANCH POINTS

From the brief discussion of the equivalent Schrödinger equation in Sec. II, it appears that in general the spectrum for the FPE (2) consists of three parts (see also [1] and Sec. III F). Discrete eigenmodes correspond to residues at isolated poles located on the rightmost part of the eigenvalue locus L [Eqs. (57) and (58)] in the complex q plane, while continuum contributions are obtained as integrals along this locus. For the continuum of reflecting states, the integral runs along (part of) a branch cut upon L, and for the free continuum the path covers the remaining part of L. The locus intersection $q=-\mu$ and the branch points mark the edges of the reflecting continuum. The branch points are necessary to define the cut. In the integral representation (52) the branch points of the integrand originate from the SK function M, which is not known explicitly. It is, however, possible to obtain an analytical expression for the branch points, in the following two ways.

1. Equivalent Schrödinger potential

The two asymptotic energy levels (19) of the Schrödinger potential (18) delimit the reflecting continuum. The q values corresponding to these eigenvalues λ are found by use of Eq. (56). One has

$$\lambda(-\infty)=\alpha\mu^2=-\alpha p_0(q_A)=\alpha[\mu^2-(q_A+\mu)^2] \quad \text{(B1)}$$

for

$$q_A=-\mu, \quad \text{(B2)}$$

which is the intersection of locus L, and

$$\lambda(+\infty)=\frac{\delta^2}{\alpha}=\alpha[\mu^2-(q_C+\mu)^2] \quad \text{(B3)}$$

for

$$q_C^{\pm}=-\mu\pm\left(\mu^2-\frac{\delta^2}{\alpha^2}\right)^{1/2}, \quad \text{(B4)}$$

which are two branch points, symmetrically located with respect to the intersection (B2) on either the horizontal ($\mu^2>\delta^2/\alpha^2$) or the vertical ($\mu^2\leqslant\delta^2/\alpha^2$) part of L.

2. Spectral invariance under reflection

The reflection symmetry of the FPE (2) as expressed by Eq. (5) does not involve the time variable. As such, the problem defined by the overbar transformed variables and parameters

$$\overline{x}=-x, \quad \overline{t}=t, \quad \overline{\alpha}=\frac{1}{\alpha}, \quad \overline{\beta}=\frac{\beta}{\alpha}, \quad \overline{\gamma}=\frac{\gamma}{\alpha},$$

$$\overline{\delta}=-\mu, \quad \overline{\epsilon}=-\epsilon, \quad \overline{\mu}=-\delta \quad \text{(B5)}$$

is isospectral with the original one. It follows from Eq. (56) that

$$-\lambda=\alpha p_0(q)=\overline{\alpha p_0}(\overline{q}) \quad \text{(B6)}$$

or, explicitly,

$$\alpha(q^2+2\mu q)=\frac{1}{\alpha}(\overline{q^2}-2\overline{\delta q}), \quad \text{(B7)}$$

which defines the transformation of the eigenvalue locus L [Eqs. (57) and (58)] in the q plane to the corresponding locus $\overline{L}$ in the $\overline{q}$ plane. Spectral separation points must preserve their role under the transformation (B7). The intersection $\overline{q}=\delta$ of the $\overline{L}$ locus, e.g., is seen to have the two images (B4) in the q plane, which hence must be the branch points there.

3. Discussion

It is instructive to trace the origin of the branch points. The SK function $M(z,p)$ is entire in z, so multivaluedness must enter via the p dependence. The latter enters solely via the zeros of the polynomials A_0, B_0, and C_0, determining the factorizer equations (40). One has explicitly

$$A_0(z,p)=-\alpha(z-a^+)(z-a^-),$$

$$a^{\pm}(p)=-\mu\pm\left(\mu^2+\frac{p}{\alpha}\right)^{1/2},$$

$$B_0(z,p)=-\beta(z-b^+)(z-b^-),$$

$$b^{\pm}(p)=-\epsilon\pm\left(\epsilon^2+\frac{\gamma p}{\beta}\right)^{1/2},$$

$$C_0(z,p)=-(z-c^+)(z-c^-),$$

$$c^{\pm}(p)=-\delta\pm(\delta^2+\alpha p)^{1/2}. \quad \text{(B8)}$$

Upon substitution of $p=\alpha p_0(q)$ [as in $M_0(q,\alpha p_0(q))$] it is seen that the above zeros become

$$a^{\pm}(\alpha p_0(q))=-\mu\pm[(q+\mu)^2]^{1/2},$$

$$b^{\pm}(\alpha p_0(q))=-\epsilon\pm\left[\epsilon^2+\frac{\alpha\gamma}{\beta}(q^2+2\mu q)\right]^{1/2},$$

$$c^{\pm}(\alpha p_0(q))=-\delta\pm[\delta^2+\alpha^2(q^2+2\mu q)]^{1/2}, \quad \text{(B9)}$$

displaying, respectively, the branch points

$$q_A^{\pm}=-\mu, \quad q_B^{\pm}=-\mu\pm\left(\mu^2-\frac{\beta\epsilon^2}{\alpha\gamma}\right)^{1/2},$$

$$q_C^{\pm}=-\mu\pm\left(\mu^2-\frac{\delta^2}{\alpha^2}\right)^{1/2}. \quad \text{(B10)}$$

As in [1], the branch points (B4) are seen to originate from the zeros in the coefficient (here C_0) of the highest power (Δ^2) of the shift operator in Eq. (24). The locus intersection is seen to correspond to a degenerate pair of branch points $q_A^{\pm}$ from A_0. Branch points $q_B^{\pm}$, likewise on locus L, do not seem to have a physical role in general (except possibly for marking the transition between the continua of penetrating and free states in barrier-well potential configurations). As the $b^{\pm}$ branches are always symmetrically represented in the ICF's (see Appendix A) it is expected that $q_B^{\pm}$ remain latent branch points indeed. They get an active role, however, when B_0 takes over from a vanishing C_0 or A_0, as in the hypergeometric limit cases (see Sec. V).

APPENDIX C: HEUN'S DIFFERENTIAL EQUATION

The spectral problem originating from separation of variables in the FPE (2) (or, more directly, in the adjoint backward Kolmogorov equation) is described by the ordinary second-order differential equation

$$(e^{2x}+\beta e^x+\alpha)\frac{d^2\varphi}{dx^2}+2(\delta e^{2x}+\beta\epsilon e^x+\alpha\mu)\frac{d\varphi}{dx}$$

$$+\lambda(\alpha e^{2x}+\gamma e^x+1)\varphi=0. \quad \text{(C1)}$$

The transformation

$$y=\frac{e^x}{\rho^+}, \quad \text{(C2)}$$

where [see Eq. (81)]

$$\rho^{\pm}=\frac{1}{2}[-\beta\pm(\beta^2-4\alpha)^{1/2}] \tag{C3}$$

are the two finite singular points of the equation, reduces Eq. (C1) to algebraic form

$$(y-1)(y-a)y^2\frac{d^2\varphi}{dy^2}+\left[(2\delta+1)y^2+\left(\frac{2\beta\epsilon}{\rho^+}-a-1\right)y+a+\frac{2\alpha\mu}{(\rho^+)^2}\right]y\frac{d\varphi}{dy}+\lambda\left(\alpha y^2+\frac{\gamma}{\rho^+}y+(\rho^+)^{-2}\right)\varphi=0,$$

$$a=\frac{\rho^-}{\rho^+}. \tag{C4}$$

Setting further

$$\varphi=y^qF(y),\quad q=-\mu\pm\left(\mu^2-\frac{\lambda}{\alpha}\right)^{1/2} \tag{C5}$$

reduces Eq. (C4) to the standard form of Heun's equation [8,19]

$$y(y-1)(y-a)F''+[C(y-1)(y-a)+Dy(y-a)+Ey(y-1)]F'+(ABy-b)F=0, \tag{C6}$$

with four regular singularities and the Riemann P symbol

$$P\begin{Bmatrix} 0 & 1 & a & \infty & & \\ 0 & 0 & 0 & A & y & b \\ 1-C & 1-D & 1-E & B & & \end{Bmatrix},$$

$$C+D+E=A+B+1. \tag{C7}$$

The nonzero exponents are defined by [see Eqs. (83) and (B8)]

$$\begin{aligned} A&=q+\delta+(\delta^2-\alpha\lambda)^{1/2}=q-c^-(-\lambda),\\ B&=q-c^+(-\lambda),\\ C&=2q+2\mu+1,\\ D&=\nu^+,\\ E&=\nu^- \end{aligned} \tag{C8}$$

and the accessory parameter b by [see Eq. (22)]

$$b=\frac{1}{\rho^+}[-\lambda\gamma-\beta(q^2+2\epsilon q)]=\frac{1}{\rho^+}B_0(q,-\lambda). \tag{C9}$$

The solutions of Eq. (C6) are discussed in [8,19].

APPENDIX D: A USEFUL INTEGRAL

Fourier transforms occurring in the present paper give rise to integrals that have the general structure

$$I=\int_{-\infty}^{\infty}dx\ e^{ax}(e^x-\rho^+)^b(e^x-\rho^-)^c. \tag{D1}$$

They are reducible ($y=e^x$) to the tabulated integral [14,15]

$$\int_0^{\infty}dy\ y^{\lambda-1}(1+\alpha y^p)^{-\mu}(1+\beta y^p)^{-\nu}=\frac{1}{p}\alpha^{-\lambda/p}\frac{\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\mu+\nu-\frac{\lambda}{p}\right)}{\Gamma(\mu+\nu)}\ {}_2F_1\left(\frac{\lambda}{p},\nu;\mu+\nu;1-\frac{\beta}{\alpha}\right),$$

$$|\arg\alpha|,|\arg\beta|<\pi,\quad p>0,\quad 0<\mathrm{Re}(\lambda)<2\ \mathrm{Re}(\mu+\nu). \tag{D2}$$

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